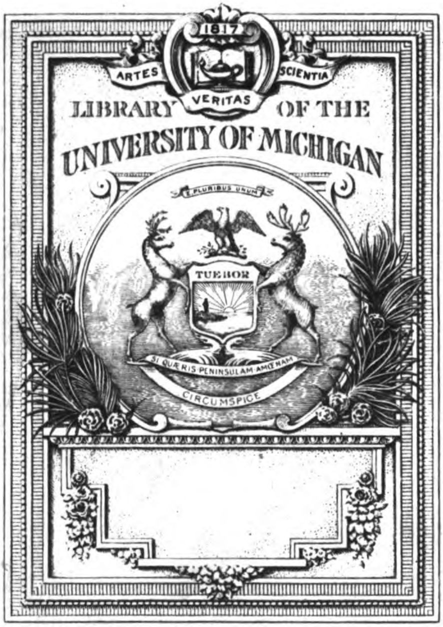
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ELEMENTS

OF

GEOMETRY.

BEING CHIEFLY

A SELECTION

FROM

PLAYFAIR'S GEOMETRY,

WITH

ADDITIONS AND IMPROVEMENTS.

FROM THE FIFTH ENGLISH EDITION.

Ed. for · Francis Nichols

PHILADELPHIA:
PRINTED AND PUBLISHED AT No. 24, ARCH STREET.
A. Walker, Agt.
1829.

EASTERN DISTRICT OF PENNSYLVANIA, to wit:

L.S. BE IT REMEMBERED, THAT on the twenty-second day of December, in the fifty-third year of the Independence of the United States of America, A. D. 1828, Francis Nichols, of the said District, hath deposited in this office the title of a book, the right whereof he claims as proprietor, in the words following, to wit:

ELEMENTS OF GEOMETRY. Being chiefly a Selection from PLAYFAIR'S GEOMETRY, with *additions* and *improvements*. From the fifth English edition.

In conformity to the Act of the Congress of the United States, entitled "An Act for the encouragement of learning, by securing the copies of maps, charts, and books, to the Authors and Proprietors of such copies, during the times therein mentioned;"—And also to the Act, entitled "An Act supplementary to an Act, entitled "An Act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies during the times therein mentioned," and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

D. CALDWELL,
Clerk of the Eastern District of Pennsylvania.

PREFACE.

Many years ago the editor of this book published two large impressions of Playfair's Geometry, with some alterations and improvements, which rendered the American edition more convenient to students than the English. The work has been long out of print, and a republication of it has been often requested by public and private teachers of geometry. It is therefore again presented to the public without an adequate compensation for the time and labour bestowed on the preparation of a new and improved edition.

The books of geometry, used in places of education in this country, are two large, and otherwise exceptionable. The best editions of Euclid's Elements are difficult and repulsive to students. They require some alterations to render them fit for the use of youth in schools and colleges. False reverence of antiquity, and old prejudice, have assigned to Euclid's Geometry a higher rank in mathematical science than its intrinsic merit can justly claim. But it is capable of alterations and improvements, and may be made a good introduction to geometry.

This book is partly an abridgment of the fifth English edition of Playfair's Geometry. It differs from Playfair's Geometry chiefly in additions, omissions, and alterations.

Many teachers of mathematics in our colleges and higher schools have often expressed a wish that a se-

lection from Euclid's Elements might be published, if it could be executed without deviating much from the form of the original. To accomplish this desirable object is the main design of the present performance.

Euclid's Geometry contains parts which are never read, with obscurities and difficulties which discourage youth, and impede the study of mathematics. The prejudices of education and custom have retained an old book in schools and colleges, to the exclusion of modern works of greater merit. We find that on the continent of Europe Euclid's Geometry has nearly sunk into oblivion, and modern systems now occupy its place. But still, as the ruins of ancient edifices furnish the materials of elegant modern structures, so Euclid's treatise of geometry may be so modified as to be converted into a plain and useful book for youth.

Some definitions, postulates, and axioms, have been added by the editor, on the supposition that they might be wanted in the course of the work, for the purpose of rendering the demonstrations of certain propositions shorter or plainer than those of Euclid. Some useful propositions, which were not known to the ancients, have been added to this treatise; and many propositions have been omitted, which are either useless, or merely auxiliary to the demonstrations of others. Some demonstrations which were long and tedious, or difficult to learners, have been rejected, and others have been adopted from modern books instead of them. But no new demonstrations have been given unless they were decidedly preferable to Euclid's.

The enunciations of many propositions are expressed with more precision and brevity than they are in former editions of Euclid's Elements. Repetitions and circumlocutions in the demonstrations have been generally avoided.

Numerous articles, marked Ed., are not found in

Playfair's Geometry, and have been taken chiefly from other books. They were collected at different times, but the editor does not recollect the names of the authors of many of them. In the latter part of the volume many references have been designedly omitted; for it appeared useless to repeat the references to preceding articles, which must have become familiar to the reader by their frequent recurrence. But the teacher should require his pupils to supply such omissions when they are reciting a demonstration.

Toward the ends of the first, third, and sixth Books there is a number of elementary propositions, which are not necessary in a course of mathematics, and may be either read or omitted. They will serve as exercises for students.

On revising the printed sheets I have observed that some changes might be made with advantage to the work. In its present form however I am confident that it will be found plainer and more convenient to students than any book of geometry which is now used in the higher places of education.

If this attempt to facilitate the study of geometry be favourably received by the public, it will be followed by another work, which is partly executed, and will contain a large collection of Geometrical Problems, Trigonometry, &c.

EDITOR.

ERRATA.

The reader is desired to attend to the following Corrections and Amendments. The editor regrets the necessity of such a long list of errors. He is not the publisher of the book, and is not responsible for typographical imperfections and errors.

PAGE.	LINE.	
21	*at bot.*	10. Two straight lines which intersect each other cannot be both parallel to the same straight line.
37	7	For 4 *Def.* read *Cor.* 3 *Def.*
38	11 *bot.*	After AF put ,
45	18	For *agles* read *angles*
54	20	For first *comma* read =
56	2	After line 1 read *AC, BD bisect each other, &*
56	10	For *angled* read *angles*
65	9	For , read .
66	2	For *contains* read *contain*
68	1	For $-\frac{1}{2} AB = \frac{1}{2} DE$ read $= \frac{1}{2} AB - \frac{1}{2} DE$
71	18	For *time* read *times*
72	14	Add, *Algebraically*. Let a denote the greater part, and x the less; then $a+x$ is the sum of the parts, and $a-x$ is the difference. Hence $(a+x) \times (a-x) = aa - xx$, which is the prop.
95	6	For *DEHG* read *DEHF*
103	6 *bot.*	For *ET* read *EI*
110	1 *bot.*	For *These* and the first four lines of p. 111, read, *Because the consequents of these two analogies are the same the antecedents are proportional (Propor. A. p. viii); therefore, the triangle ECH : ADL : : base CH : DL.*
114	22	For *meet* read *meet, for if not, the angles B and E would be equal to two right angles (29. 1)*
116	2 *bot.*	For *; wherefore* read *, and*
123	18	For 61 read 6
137	4 *bot.*	For 2 read 3
142	12	For *circles* read *circle*
149	8 *bot.*	For 2 read 3
151	2	At the upper right corner of the fig. write *E*
152	7,10,13,23	For 2 *Sup.* read 11
158	13	For *axis* read *axes*
161	15	For 11 read 12
163	25	For *p* read *n*
163	4 *bot.*	After *cylinders* read *of equal altitudes*
169	4	For 13 12 read 11 12
175	3 *bot.*	For 17 read 16

A. If the antecedents in two proportions be the same, the consequents are proportional; and if the consequents be the same, the antecedents are proportional.

If $A : B :: C : D$,
and $A : E :: C : F$;
then $B : E :: D : F$.
For $A : C :: B : D$ (36), and $A : C :: E : F$;
$\therefore B : D :: E : F$ (34), $\therefore B : E :: D : F$.
Again, if $B : A :: D : C$,
and $E : A :: F : C$;
then $B : E :: D : F$.
The proof is the same as the first.

B If the means in two proportions be the same, the extremes are proportional in a cross order; and if the extremes be the same, the means are proportional in a cross order.

$$\left.\begin{array}{l}\text{If } A : B :: C : D \\ \text{and } E : B :: C : F\end{array}\right\}, \text{ then } A : F :: E : D.$$

For $AD = BC$, and $EF = BC$ (26); $\therefore$
$AD = EF$, $\therefore A : F :: E : D$ (33).
Again, if $B : A :: D : C$,
and $B : E :: F : C$,
it may be proved in the same manner that $A : F :: E : D$.

INTRODUCTION.

OF PROPORTION.

The doctrine of Proportion, in the Fifth Book of Euclid's Elements, is obscure, and unintelligible to most readers. It is not taught either in foreign or American colleges, and is now become obsolete. It has therefore been omitted in this edition of Euclid's Elements, and a different method of treating Proportion has been substituted for it. This is the common algebraical method, which is concise, simple, and perspicuous; and is sufficient for all useful purposes in practical mathematics. The method is clear and intelligible to all persons who know the first principles of algebra. The rudiments of algebra ought to be taught before geometry, because algebra may be applied to geometry in certain cases, and facilitates the study of it.

Those persons who desire to see the doctrine of Proportion treated according to a general method which is plainer than Euclid's, and equally accurate, may consult the geometry of Playfair, Ingram, Leslie, Cresswell, and J. R. Young. Hutton, and other recent writers have adopted the algebraical method in their elements of geometry. Proportion is not properly a geometrical subject.

PROPORTION.

1. A less magnitude is said to be a part of a greater magnitude, when the less measures the greater, that is, when the less is contained in the greater a certain number of times exactly.

2. A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

Thus, if A be exactly three times B, then A is said to be a multiple of B; and B is said to be a part of A.

3. When several magnitudes are multiples of as many other magnitudes, and each magnitude contains its part the same number of times, the former magnitudes are said to be equimultiples of the latter, and the latter are said to be like parts of the former.

Thus, if A be triple of B, and C triple of D, then A, C are called equimultiples of B, D; and B, D are called like parts of A, C.

4. Two magnitudes are said to be homogeneous, or of the same kind, when the less can be multiplied so as to exceed the greater.

Thus, a minute may be multiplied till the product exceed an hour, a yard till the product exceed a mile, &c.

5. Two quantities are said to be commensurable, when they are divisible by a third quantity without a remainder; and the third quantity is called their common measure. Thus, 4 and 6 are commensurable, and 2 is their common measure.

6. Two quantities are said to be incommensurable, when they are not divisible by a third quantity without a remainder.

Thus, 4 and 7 are not commensurable, because they cannot be divided by a third number without a remainder.

7. Between any two finite quantities of the *same kind* there subsists a certain relation in respect of magnitude, which is called their *ratio.*

8. When we observe two quantities, one of which is double of the other, we acquire the idea of a particular ratio, or relation, which the greater has to the less; and when we afterward find two other quantities, one of which is also double of the other, we say that they have the same ratio which the two for-

mer quantities have to each other, or that the four quantities are proportionals. In like manner, by observing one quantity to be triple, quadruple, or any multiple of another, we acquire ideas of other ratios; and thus we obtain ideas of all ratios where the greater quantity is a multiple of the less.

9. The comparison between two quantities is made by considering how often one quantity contains the other, or how often one quantity is contained in the other. Thus, in comparing 6 with 3, we observe that 6 has a certain magnitude with respect to 3, for 6 contains 3 twice; and in comparing 6 with 2, we observe that 6 has a different relative magnitude, for 6 contains 2 three times.

10. The quantities must be of the same kind, else they cannot be compared together, and therefore no judgment of their equality or inequality can be formed. Thus, 2 hours and 3 yards cannot be compared together, because they are quantities of a different nature.

11. The ratio of A to B is expressed by $\frac{A}{B}$, or by two points placed between them, as A : B. The former quantity A is called the *antecedent* of the ratio, and the latter B the *consequent*.

12. The antecedent and consequent are called the *terms* of the ratio, and the quotient of the two terms is called the *measure, index,* or *exponent* of the ratio. Thus, if $\frac{A}{B}=m$, then m is called the *measure*, &c. of the ratio of A to B.

13. One ratio is greater than another when its antecedent is a greater multiple, part, or parts of its consequent than the antecedent of the other ratio is of its consequent. Thus, the ratio 7 : 4 is greater than the ratio 8 : 5, because $\frac{7}{4}$ and $\frac{8}{5}$ reduced to a common denominator are $\frac{35}{20}$ and $\frac{32}{20}$, and $\frac{35}{20}$ is greater than $\frac{32}{20}$ by $\frac{3}{20}$.

14. If the antecedents of any ratios be multiplied together, and also the consequents, a new ratio results, which is said to be compounded of the former ratios. Thus, if A : B and C : D be two ratios, then AC : BD is said to be compounded of the two ratios A : B and C : D.

15. If a ratio be compounded of two equal ratios, it is called a *duplicate* ratio; if of three equal ratios, it is called a *triplicate* ratio, &c. Thus, if $\frac{A}{B}=m$, and $\frac{C}{D}=m$, then $\frac{AC}{BD}=m^2$, that is, the ratio of AC to BD is duplicate of the ratio of A to

B, or of C to D. If $\frac{A}{B}=m$, $\frac{C}{D}=m$, $\frac{E}{F}=m$; then $\frac{ACE}{BDF}=m^3$, that is, the ratio of ACE to BDF is triplicate of the ratio of A to B, or of C to D, or of E to F.

16. The ratio of A to B is said to be one third of the ratio of A^3 to B^3, and the ratio of $A^{\frac{1}{m}}$ to $B^{\frac{1}{m}}$ is said to be an mth part of the ratio of A to B.

17. Let the first ratio be $a:1$, then $a^2:1$, $a^3:1$, ... $a^n:1$, are twice, thrice, ... n times the first ratio $a:1$. n, the index of a, shows what multiple or part of the ratio $a^n:1$ the first ratio is. For this reason the indices 1, 2, 3, ... n are called measures of the ratios $a^1:1$, $a^2:1$, $a^3:1$, ..., $a^n:1$.

18. Proportion is an equality of ratios.

Thus, let $\frac{A}{B}=m$, and $\frac{C}{D}=n$; then, if $m=n$, the two ratios are equal, that is, A has the same ratio to B which C has to D.

If m be greater than n, then A has to B a greater ratio than C has to D, and the four quantities are not proportional.

If m be less than n, then A has to B a less ratio than C has to D, and the four quantities are not proportional.

A proportion is thus expressed, A : B : : C : D, or A : B = C : D, or $\frac{A}{B}=\frac{C}{D}$, and is read A is to B as C is to D.

19. Hence four quantities are proportional when the first contains the second as often as the third contains the fourth.

Note. It will not always happen that the first quantity contains the second exactly, or the third contains the fourth; but the criterion of proportion is complete if the two fractions which consist of the terms of the two ratios be equal; that is, when $\frac{A}{B}=\frac{C}{D}$, then A : B : : C : D.

If A = 10, B = 5, C = 8, D = 4; then $\frac{A}{B}=\frac{10}{5}=2$, and $\frac{C}{D}=\frac{8}{4}=2$. Hence 10 : 5 : : 8 : 4. Again, 3 : 4 : : 9 : 12, for $\frac{3}{4}=\frac{9}{12}$.

The following is a general definition of proportion, whether the terms of the two ratios be commensurable or incommensurable.

20. Four quantities are proportional when any multiple of the first contains the second as often as the same multiple of the third contains the fourth.

Let A, B, C, D be four quantities, and m any number, and let $\frac{mA}{B}=\frac{mC}{D}$; then A : B : : C : D. Let A = 2, B = 3, C = 4, D = 6, and m = 3; then $\frac{mA}{B}=\frac{6}{3}=2$, and $\frac{mC}{D}=\frac{12}{6}=2$; therefore 2 : 3 : : 4 : 6.

21. The terms A and D are called extremes, and the terms B and C are called means.

22. In any proportion the two antecedents, or the two consequents, are sometimes called homologous terms; and each antecedent with its consequent are called analogous terms.

23. If the sum or difference of two numbers be multiplied by any number, the product is equal to the sum or difference of the separate products of the first two numbers multiplied by the third.

Let A, B, C, be three numbers; (B+C) A=AB+AC, and (B—C) A=AB—AC.

For the product AB is the same as each unit in B repeated A times, and the product AC is the same as each unit in C repeated A times; therefore the sum of the products AB+AC is equal to the units contained in B+C repeated A times, or AB +AC is equal to the sum of the numbers B and C multiplied by A.

Again, for the same reason, the difference between the products AB and AC, or AB—AC, must be equal to the difference of the units contained in B and C, or in B—C, repeated A times; that is, AB—AC is equal to the difference of the numbers B and C multiplied by A.

24. Cor. 1. Hence a number which measures (or divides) any two numbers will measure their sum and difference. Thus, A measures AB and AC, and also AB+AC and AB—AC.

25. Cor. 2. Hence it is manifest that the first part of the proposition may be extended to more than two numbers, or that AB+AC+AD+ &c. = A (B+C+D+ &c.)

26. If four quantities be proportional, the product of the extremes is equal to the product of the means.

If $A : B :: C : D$, then $AD = BC$.

For $\frac{A}{B} = \frac{C}{D}, \therefore AD = BC$.

27. If the first quantity be to the second as the second to the third, the product of the extremes will be equal to the square of the mean.

If $A : B :: B : C$, then $AC = B^2$.

For $\frac{A}{B} = \frac{B}{C}, \therefore AC = B^2$.

28. Cor. $B = \sqrt{AC}$, that is, a geometrical mean proportional between two quantities is equal to the square root of their product.

29. If any three terms of a proportion be given, the fourth term may be found.

Let x be the unknown term, and let $A : B :: C : x$, then $Ax = BC$ (26), $\therefore x = \frac{BC}{A}$. Again, let $A : B :: x : D$, then $Bx = AD$, $\therefore x = \frac{AD}{B}$.

NOTE. This article contains the demonstration of the Rule of Three in Arithmetic.

30. Equimultiples of any quantities have the same ratio to one another which the quantities have; and like parts of any quantities have the same ratio to one another which the quantities have.

Let A and B be any quantities, and m any number; $mA : mB :: A : B$, and $\frac{1}{m}A : \frac{1}{m}B :: A : B$.

For $A : B :: A : B, \therefore \frac{A}{B} = \frac{A}{B}, \therefore \frac{mA}{mB} = \frac{A}{B}$, or $mA : mB :: A : B$.

Again, because $A : B :: A : B, \frac{A}{B} = \frac{A}{B}, \therefore \frac{\frac{1}{m}A}{\frac{1}{m}B} = \frac{A}{B}$, or $\frac{1}{m}A : \frac{1}{m}B :: A : B$.

31. If four quantities be proportional, according as the first quantity is greater than, equal to, or less than the second, the third quantity is greater than, equal to, or less than the fourth.

Let $A : B :: C : D$, then $AD = BC$ (26), $\therefore$ if A be greater than B, then C is greater than D, if $A=B$, then $C=D$, and if A be less than B, then C is less than D.

32. If four quantities be proportional, according as the first quantity is greater than, equal to, or less than the third, the second quantity is greater than, equal to, or less than the fourth.

Let $A : B :: C : D$, then $AD = BC$ (26), $\therefore$ if A be greater than C, then B is greater than D, if $A=C$, then $B=D$, and if A be less than C, then B is less than D.

33. If the product of two quantities be equal to the product of two other quantities, these four quantities may be turned into a proportion by making the terms of one product the means, and the terms of the other product the extremes.

Let $AD=BC$, then $\frac{A}{B} = \frac{C}{D}$, that is, $A : B :: C : D$.

34. Quantities which have the same ratio to the same qeantities are proportional.

Let $A : B :: C : D$, and $C : D :: E : F$, then

$A : B :: E : F$. Because $\frac{A}{B} = \frac{C}{D}$, and $\frac{C}{D} = \frac{E}{F}$, $\frac{A}{B} = \frac{E}{F}$,

or $A : B :: E : F$.

35. If four quantities be proportional, they are also proportional when taken inversely.

If $A : B :: C : D$, then $B : A :: D : C$. For $\frac{A}{B} = \frac{C}{D}$, $\therefore \frac{B}{A} = \frac{D}{C}$, or $B : A :: D : C$.

36. If four quantities be proportional, they are also proportional when taken alternately.

If $A : B :: C : D$, then (by alternation) $A : C :: B : D$. For $AD = BC$ (26), $\therefore A : C :: B : D$ (33).

37. If four quantities be proportional, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.

Let $A : B :: C : D$, then (by composition) $A + B : B :: C + D : D$.

For $\frac{A}{B} = \frac{C}{D}$, and $\frac{B}{B} = \frac{D}{D}$; $\therefore \frac{A + B}{B} = \frac{C + D}{D}$, or $A + B : B :: C + D : D$.

38. If four quantities be proportional, the difference between the first and second is to the second, as the difference between the third and fourth is to the fourth.

Let $A : B :: C : D$, then (by division) $A - B : B :: C - D : D$.

For $\frac{A}{B} = \frac{C}{D}$, and $\frac{B}{B} = \frac{D}{D}$; $\therefore \frac{A - B}{B} = \frac{C - D}{D}$, or $A - B : B :: C - D : D$.

39. If four quantities be proportional, the first is to the sum or difference of the first and second, as the third is to the sum or difference of the third and fourth.

Let $A : B :: C : D$, then (by conversion) $A : A + B :: C : C + D$, and $A : A - B :: C : C - D$.

For $B : A :: D : C$ (35), or $\frac{B}{A} = \frac{D}{C}$, and $\frac{A}{A} = \frac{C}{C}$; $\therefore$ $\frac{B \pm A}{A} = \frac{D \pm C}{C}$, $\therefore B \pm A : A :: D \pm C : C$,

$\therefore A : B \pm A :: C : D \pm C$.

Cor. $B \pm A : A :: D \pm C : C$ (35).

40. If four quantities be proportional, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

Let $A : B :: C : D$, then (by mixing)

$$A + B : A - B :: C + D : C - D.$$

For $A + B : B :: C + D : D$ (37),
$\therefore A + B : C + D :: B : D$ (36).

Again, $A - B : B :: C - D : D$ (38),
$\therefore A - B : C - D :: B : D$.

Hence $A + B : C + D :: A - B : C - D$ (34)
$\therefore A + B : A - B :: C + D : C - D$ (36).

41. If several quantities be proportional, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $A : B :: C : D :: E : F :: G : H$, &c. that is, let
$A : B :: C : D$, and $A : B :: E : F$, &c.
then $A : B :: A + C + E + G : B + D + F + H$.
For $AD = BC$ (26), $AF = BE$, $AH = BG$; also $AB = BA$; $\therefore$
$AB + AD + AF + AH = BA + BC + BE + BG$, or
$A \times (B + D + F + H) = B \times (A + C + E + G)$ (25), $\therefore$
$A : B :: A + C + E + G : B + D + F + H$ (33).

42. If there be several ranks of proportional quantities, the products of the corresponding terms will be proportional.

If $A : B :: C : D$
and $E : F :: G : H$
and $I : K :: L : M$
then $AEI : BFK :: CGL : DHM$.

For $\frac{A}{B} = \frac{C}{D}$, & $\frac{E}{F} = \frac{G}{H}$, & $\frac{I}{K} = \frac{L}{M}$; $\therefore \frac{AEI}{BFK} = \frac{CGL}{DHM}$,
$\therefore AEI : BFK :: CGL : DHM$.

43. If the same quantity occur in all the terms of a compound proportion, or in either of its two ratios, that quantity may be rejected, and the remaining terms will be proportional.

If $A : B :: C : D$
and $E : A :: F : C$
then $AE : AB :: CF : CD$ (42)
$\therefore E : B :: F : D$ (30).
Again, let $A : B :: C : D$
and $B : E :: D : F$
and $E : G :: F : H$
then $ABE : BEG :: CDF : DFH$
$\therefore A : G :: C : H$.

44. If four quantities be proportional, the like powers or roots of those quantities will be proportional.

If $A : B :: C : D$, then $A^n : B^n :: C^n : D^n$, n being either a whole number or a fraction.

Because $\frac{A}{B} = \frac{C}{D}$, $\frac{A^2}{B^2} = \frac{C^2}{D^2}$, or $\frac{A^{\frac{1}{2}}}{B^{\frac{1}{2}}} = \frac{C^{\frac{1}{2}}}{D^{\frac{1}{2}}}$,

and $\frac{A^n}{B^n} = \frac{C^n}{D^n}$, that is, $A^n : B^n :: C^n : D^n$.

45. If several quantities, A, B, C, D, E, &c. be in continued proportion, the first quantity will be to the third as the square of the first is to the square of the second; the first quantity will be to the fourth as the cube of the first is to the cube of the second, &c. that is,

$A : C :: A^2 : B^2$, or in the duplicate ratio of $A : B$;
$A : D :: A^3 : B^3$, or in the triplicate ratio of $A : B$;
$A : E :: A^4 : B^4$, or in the quadruplicate ratio of $A : B$; &c. &c.

1. Let $A : B :: B : C$, then $B^2 = AC$ (26), $\therefore$
$A \times B^2 = A \times AC = A^2 \times C$, $\therefore A : C :: A^2 : B^2$ (33).

2. Let $A : B :: B : C :: C : D$, then $A : C :: A^2 : B^2$ (as above).
But $C : D :: A : B$
$\therefore A : D :: A^3 : B^3$ (42 & 43).

3. Let $A : B :: B : C :: C : D :: D : E$,
then $A : D :: A^3 : B^3$ (as above).
But $D : E :: A : B$
$\therefore A : E :: A^4 : B^4$.

Scholium. The doctrine of Proportion in this tract is more general than that in Euclid's Elements. It includes the properties of both proportional numbers and of magnitudes. Euclid's Fifth Book contains the properties of proportional magnitudes only. The word *quantity*, employed above, denotes both numbers and magnitudes, or objects having extension.

This tract is chiefly taken from the treatises of Algebra of Wood and Bridge, and is the same in substance as is taught in foreign universities, instead of the Fifth Book of Euclid.

THE MOST USEFUL PROPERTIES OF PROPORTION.

If four quantities be proportional, the product of the extremes will be equal to the product of the means.

If the first quantity be to the second as the second to the third, the product of the extremes will be equal to the square of the mean.

If four quantities be proportional, according as the first quantity is equal to, greater, or less than the second, the third is equal to, greater, or less than the fourth; or according as the first quantity is equal to, greater, or less than the third, the second is equal to, greater, or less than the fourth.

If the product of two quantities be equal to the product of two other quantities, these four quantities may be turned into a proportion by making the terms of one product the means, and the terms of the other product the extremes.

Quantities which have the same ratio to the same quantities are proportional.

If four quantities be proportional, they are also proportional by inversion, alternation, composition, division, conversion, and mixing.

If several quantities be proportional, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

If four quantities be proportional, and if the first and second be multiplied or divided by any quantity, and also the third and fourth, the resulting quantities will be proportional; or if the first and third be multiplied or divided by any quantity, and also the second and fourth, the resulting quantities will be proportional.

If there be several ranks of proportional quantities, the products of the corresponding terms will be proportional.

If four quantities be proportional, the like powers, and the like roots of those quantities will be proportional.

If three quantities be proportional, the first quantity will be to the third, as the square of the first is to the square of the second.

ELEMENTS OF GEOMETRY.

BOOK I.*

EXPLANATION OF TERMS. Ed.

A PROPOSITION is something which is proposed either to be done, or to be demonstrated; and is either a theorem or a problem.

The words in which a proposition is expressed are called the *enunciation* of the proposition.

A *theorem* is something which is proposed to be demonstrated; or, it is a truth which becomes evident by means of a train of reasoning called a demonstration.

A *problem* is something which is proposed to be done; or, it is a question proposed, which requires a solution.

A *lemma* is something which is premised, or previously demonstrated, with a design to facilitate the demonstration of a theorem, or the solution of a problem.

A *corollary* is a consequent truth, or proposition deduced immediately from some preceding truth, proposition, or demonstration.

A *scholium* is a remark made upon one or more preceding propositions, and tending to show their connection, or restriction, or extension, or utility.

An *axiom* is a self evident proposition.

A *postulate* is something required to be done, which is so easy and evident that its practicability cannot be denied.

* *See Notes at the end of the volume.*

DEFINITIONS.

A. GEOMETRY is that science which treats of the properties and relations of space; or, it is that science whose object is the measure of extension. EDITOR.

B. Extension has three dimensions, length, breadth, and thickness. ED.

C. Magnitude is that kind of quantity which we conceive to be extended, and divisible into parts. There are three sorts of magnitudes, a line, a surface, and a solid. ED.

1. A Point is that which has position, but not magnitude.

2. A Line is length without breadth.

Corollary. The extremities of a line are points; and the intersection of two lines is a point.

3. A straight line is that which every where tends the same way. *See Note.* ED.

4. *Cor.* Hence a straight line is the least distance between two points. ED.

5. A curve line is that which continually changes its direction between its extreme points. ED.

6. A superficies, or surface, is that magnitude which has only length and breadth.

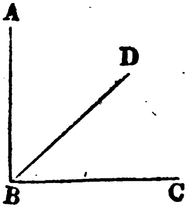
7. *Cor.* The extremities of a superficies are lines; and the intersection of one superficies with another is a line.

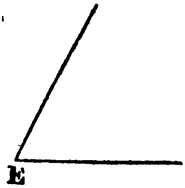
8. A plane superficies is that in which any two poins being taken, the straight line which joins them lies wholly in that superficies.

9. If two straight lines diverge from the same point, the opening between them is called an angle. ED.

10. An angle is formed by the meeting, or intersection of two lines. The point of concourse of the two lines is called the summit, vertex, or angular point. ED.

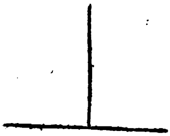
The magnitude of an angle does not depend on the length of the two lines which form it, but on the wideness of their opening. Thus, the angle ABC is greater than the angle DBC. ED.





When several angles are at one point B, any one of them is expressed by three letters, of which the letter at the vertex of the angle is put between the other two letters, and one of these two is somewhere on one of those straight lines, and the other on the other line: thus, the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, BD, is named the angle ABD, or DBA; and that which is contained by BD, CB, is called the angle DBC, or CBD. If there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle at E.

10. When a straight line standing on another straight line makes the adjacent angles equal to each other, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



Otherwise. When a straight line standing on another straight line does not incline to either side of it, the former line is said to be perpendicular to the latter; and the two angles which it forms with the latter are called right angles. Ed.

11. *Cor.* Hence all right angles are equal to one another. For they are formed by straight lines standing perpendicularly on other straight lines; and since they are formed exactly in the same manner they are necessarily equal to one another, Ed.

12. An obtuse angle is that which is greater than a right angle.

13. An acute angle is that which is less than a right angle.

Note. An angle which is either acute or obtuse is often called an oblique angle. ED.

14. If two lines be in the same plane, and do not meet, though produced ever so far both ways, they are called parallel lines.

Otherwise. If two lines be in the same plane, and have no inclination to each other, they are called parallel lines. ED.

15. *Cor.* 1. Hence parallel lines are equidistant. For they neither accede to, nor recede from each other. ED.

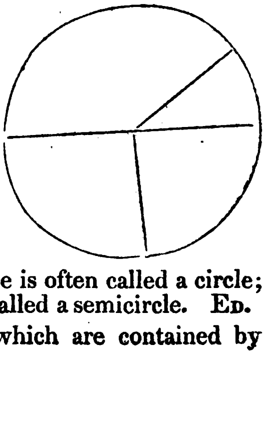
16. *Cor.* 2. If two straight lines be perpendicular to the same straight line, they are parallel to each other. For they have no inclination to each other. ED.

17. A figure is that which is enclosed by one or more boundaries.

18. If a finite straight line be supposed to revolve in the same plane, about one of its extremities, which is fixed, until it arrive at the place from which it began to move, the surface described by the revolving line is called a circle. ED.

19. The fixed point about which the line revolves is called the centre of the circle; the revolving line is called a radius; and the curve line described by the moveable extremity of the revolving line is called the circumference of the circle. ED.

20. *Cor.* Hence a circle is a plane figure contained by a curve line, or circumference, and all straight lines drawn from the centre to the circumference are equal to one another.



21. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference; and a radius is a straight line drawn from the centre to the circumference.

22. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

Note. The circumference of a circle is often called a circle; and the arch of a semicircle is often called a semicircle. ED.

23. Rectilineal figures are those which are contained by straight lines.

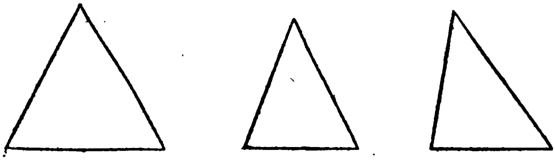
24. Trilateral figures, or triangles, are contained by three straight lines.

25. Quadrilateral figures by four straight lines.

26. Multilateral figures, or polygons, by more than four straight lines.

27. Of three-sided figures, an equilateral triangle is that which has three equal sides.

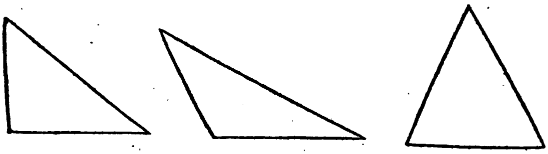
28. An isosceles triangle is that which has two sides equal.



29. A scalene triangle is that which has three unequal sides.

30. A right angled triangle is that which has a right angle.

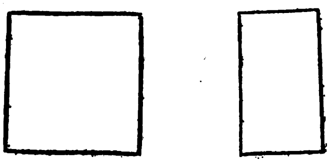
31. An obtuse angled triangle is that which has an obtuse angle.



32. An acute angled triangle is that which has three acute angles.

Note. A triangle which has no right angle is often called an oblique angled triangle. ED.

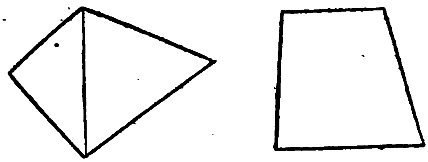
33. Of quadrilateral figures, a square is that which has one right angle, and all its sides equal.



34. A rectangle is that which has all its angles right angles, but has not all its sides equal.

35. A parallelogram is that which has its two opposite sides parallel.

36. A trapezium is a four-sided figure, of which the opposite sides are not parallel; and the diagonal is the straight line joining two of its opposite angles. ED.



37. A straight line joining two opposite angles of any quadrilateral figure, or two opposite angles of any polygon, is called a diagonal. ED.

38. In a right angled triangle the side opposite to the right angle is called the hypothenuse; and of the other two sides one is called the base and the other the perpendicular, according to the position in which the triangle is described. Also, the two sides are sometimes called the legs. ED.

39. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

40. That part of Geometry which treats of the measurement and position of plane figures, of straight lines, and of rectilineal angles, is called Plane Geometry.

POSTULATES.

1. LET it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. That a circle may be described from any centre, at any distance from that centre.

4. That a straight line which meets one of two parallel straight lines may be produced till it meet the other. ED.

5. If there be two equal straight lines, and if any figure whatever be constituted on one of them, a figure exactly similar to it may be constituted on the other.

AXIOMS.

1. Things which are equal to the same thing are equal to one another. Also things which are equal to equal things are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are doubles of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

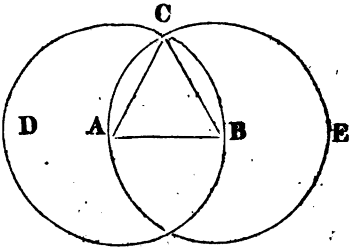
A. The whole is equal to all its parts taken together. Ed.

B. It is impossible for the same thing to have, at the same time, two qualities which are inconsistent with each other. Ed.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle on a given finite straight line. *See Note.*

Let AB be the given straight line; it is required to describe an equilateral triangle on it.



From the centre A, at the distance AB, describe the circumference of the circle BCD (3 Postulate); and from the centre B, at the distance BA, describe the circumference of the circle ACE. Then, because the circumference of each circle passes through the centre of the other, the two circles will cut each other in some point C. From the point C draw the straight lines CA, CB, to the points A, B (1 Post.); ABC will be an equilateral triangle.

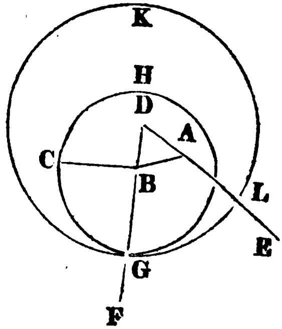
Because the point A is the centre of the circle BCD, AC is equal to AB (20 Definition); and because the point B is the centre of the circle ACE, BC is equal to BA: therefore CA and CB are each of them equal to AB. But things which are equal to the same thing are equal to one another (1 Axiom); therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and they form a triangle ABC; therefore the triangle ABC is equilateral; and it is described on the given straight line AB. Which was required to be done.*

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw from A a straight line equal to BC.

* This demonstration is full and minute.

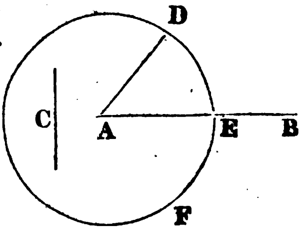


Join the points A, B, and on AB describe the equilateral triangle DAB (Prop. 1). From the centre B, at the distance BC, describe the circle CGH (3 Post.), and produce DB to G (2 Post.). From the centre D, at the distance DG, describe the circle GKL, and produce DA to L. AL will be equal to BC.

Because the point B is the centre of the circle CGH, BC is equal to BG (20 Def.); and because D is the centre of the circle GKL, DL is equal to DG. But DA, BD, parts of them, are equal (27 Def.); therefore the remainder AL is equal to the remainder BG (3 Ax.). But BC is equal to BG; wherefore AL and BC are each of them equal to BG; therefore the line AL is equal to BC (1 Ax.); and AL is drawn from the given point A, as was to be done.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be two given straight lines, whereof AB is the greater. It is required to cut off from AB a part equal to C.

From the point A draw the straight line AD equal to C (Prop. 2.); and from the centre A, at the distance AD, describe the circle DEF, cutting AB in E (3 Post.). Then AE is equal to C.

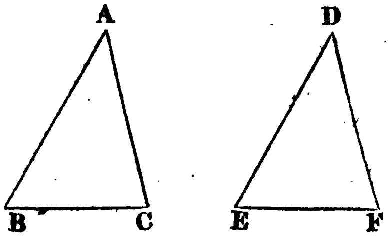
Because A is the centre of the circle DEF, AE is equal to AD (20 Def.). But the straight line C is equal to AD; whence AE and C are each of them equal to AD; wherefore AE is equal to C (1 Ax.). Therefore, from AB, the greater of two straight lines, a part AE has been cut off equal to C, the less. Which was to be done.

PROPOSITION IV. THEOREM.

If two triangles have two sides of one triangle equal to two sides of the other, each to each; and have also the angles contained by those sides equal to each

other; their third sides will be equal; and their surfaces will be equal; and their other angles to which the equal sides are opposite will be equal, each to each.*

Let ABC, DEF be two triangles, which have the side AB equal to DE, and AC to DF; and the angle A equal to D; then the base BC will be equal to EF, and the triangle ABC to DEF; and the other angles, to which the equal sides are opposite, will be equal, each to each, namely, the angle B to E, and the angle C to F.



If the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the straight line AB on DE, the point B will coincide with E, because AB is equal to DE; and because AB coincides with DE, and the angle A is equal to D, AC will coincide with DF; wherefore also the point C will coincide with F, because AC is equal to DF. But the point B coincides with E; wherefore the base BC will coincide with EF, and will be equal to it. Therefore the whole triangle ABC will coincide with the whole triangle DEF, and be equal to it; and the remaining angles of one triangle will coincide with the remaining angles of the other, and be equal to them, namely, the angle B to E, and the angle C to F.

Therefore, if two triangles have two sides of one triangle equal to two sides of the other, each to each, and have also the angles contained by those sides equal to each other; their bases will be equal, and their surfaces will be equal, and their other angles to which the equal sides are opposite will be equal, each to each. Which was to be demonstrated.

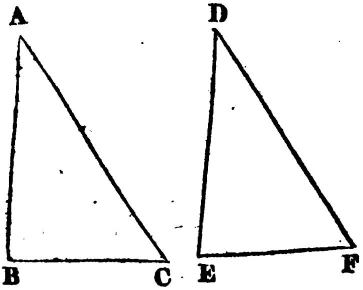
**Shorter enunciation, thus:*—If two sides of one triangle be equal to two sides of another, each to each, and if the angles contained by those sides be also equal, the triangles will be equal in all respects.

PROPOSITION A. THEOREM.

This is Prop. 26.

If two triangles have two angles of one triangle equal to two angles of the other, each to each, and have also the sides between those angles equal to each other; then will the other two sides be equal, each to each, and the third angle of one triangle to the third angle of the other.

Let ABC, DEF be two triangles which have the angles B, C equal to the angles E, F, each to each, and the side BC equal to EF; then the other sides will be equal, each to each, namely, AB to DE, and AC to DF; and the third angle A to the third angle D.



Let the triangle ABC be applied to DEF, so that the side BC may coincide with EF; then the side AB will lie on DE, because the angle B is equal to E, and the side AC will lie on DF, because the angle C is equal to F. Therefore the point A where BA, CA meet, will fall on the angular point D. Consequently the side AB coincides with DE, AC with DF, the angle A with D, and the triangle ABC with DEF. Therefore, if two triangles &c. Q. E. D. ED.

Cor. 1. if two angles of a triangle be equal to each other, the sides opposite to them are equal.

For suppose the two angles at the base of each triangle to become equal to each other, then the four angles at the bases of both triangles will be equal to one another; therefore the sides opposite to them will be equal to one another by the proposition. ED.

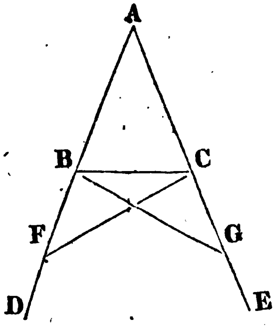
Cor. 2. If two triangles be mutually equiangular, and have two corresponding sides equal to each other, the other corresponding sides will be equal, and the triangles will be equal in all respects. Ed.

PROPOSITION V. THEOREM.

IF two sides of a triangle be equal, the angles opposite to them are equal.

Let ABC be a triangle, of which the side AB is equal to AC; the angle ABC is equal to ACB.

Produce AB, AC to D and E. In BD take any point F, and from AE, the greater, cut off AG equal to AF, the less (3. 1.), and join FC, GB.



The side AF is equal to AG, and AB to AC, and the angle A is common to the two triangles AFC, AGB; therefore the base FC is equal to GB (4. 1.), and the angle ACF to ABG, and the angle AFC to AGB.

Because AF is equal to AG, and the part AB to AC, the remainder BF is equal to the remainder CG (3 Ax.)

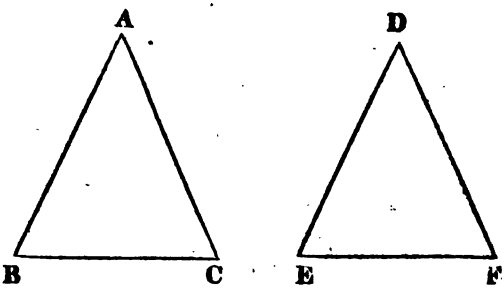
In the triangles BFC, CGB, the side BF is equal to CG, and FC to GB, and the angle BFC to CGB; therefore the angle BCF is equal to CBG (4. 1.).

Now, since it has been proved that the angle ABG is equal to ACF, and the part CBG to the part BCF, the remaining angle ABC is equal to the remaining angle ACB (3 Ax.). Therefore, the angles at the base &c. Q. E. D.*

Otherwise. If the side AB be equal to AC, the angle ABC is equal to ACB. For no reason can be assigned that the angle ABC should be either greater or less than ACB; therefore the angles ABC, ACB are equal. Q. E. D. Ed.

Otherwise. This proposition is merely a corollary to the fourth.

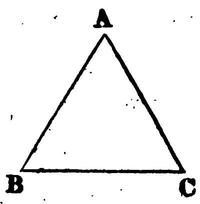
**Quod erat demonstrandum*, which was to be demonstrated.



For, let the sides AB, AC be equal to each other, and also the sides DE, DF be equal; then the four sides AB, AC, DE, DF, are equal to one another; therefore the four angles subtended by those four sides are equal to one another (4, 1.). Therefore the two angles at the base of each triangle are equal to each other. Q. E. D. Ed.

Cor. Hence every equilateral triangle is also equiangular.

Let ABC be an equilateral triangle; then because the two sides AB, AC are equal, the angles B and C at the base are equal. Again, because the sides BA, BC are equal, the angles A and C at the base are equal. Hence each of the angles B and A is equal to C; therefore they are equal to each other (1 Ax.). Wherefore the three angles of the triangle ABC are equal, that is, the triangle ABC is equiangular. Ed.

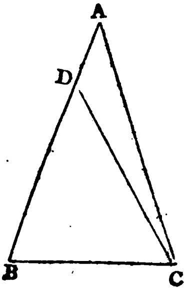


PROPOSITION VI. THEOREM.*

If two angles of a triangle be equal, the sides opposite to them are equal. *See Note.*

Let ABC be a triangle having the angle B equal to C; the side AB is also equal to AC.

For if AB be not equal to AC, one of them is greater than the other. Let AB be the greater, and from it cut off DB equal to AC, the less (3. 1.), and join DC. In the triangles DBC, ACB, DB is equal to AC, and BC common to both, and the angle B is equal to ACB (by the supposition); therefore the triangle DBC is equal to ACB (4. 1.), the less to the greater, which is absurd. Therefore the side AB is not greater than AC. In the same manner it may be proved that AB is not less than AC. Consequently AB is equal to AC. Therefore, if two angles &c. Q. E. D.



Otherwise. If the angle B be equal to C, the side AB is equal to AC. For no reason can be assigned that the side AB should be either greater or less than AC; therefore the sides AB AC are equal. Q. E. D. Ed.

Cor. Hence every equiangular triangle is also equilateral.

Let ABC be an equiangular triangle (figure to Cor. Prop. V); then because the angle B is equal to C, the side AC is equal to AB; and because the angle A is equal to C, the side BC is equal to AB. Hence each of the sides AC and BC is equal to AB; therefore they are equal to each other (1 Ax.). Therefore the three sides of the triangle ABC are equal, that is the triangle ABC is equilateral. Ed.

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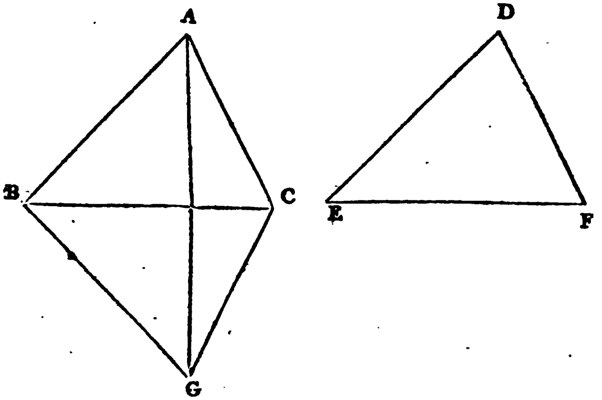
* This is the converse of Proposition V, and may be inferred from it without demonstration.

PROPOSITION VIII. THEOREM

If two triangles have the three sides of one triangle equal to the three sides of the other, each to each, they are equal in all respects.

Let ABC, DEF be two triangles having the side AB equal to DE, and AC to DF, and BC to EF; the triangle ABC is equal to DEF, the angle A to D, the angle B to E, and the angle C to F.



Let the triangle DEF be applied to the triangle ABC, so that their longest sides, BC, EF, may coincide; and let BGC represent the triangle DEF in an inverted position. Join AG. Because the sides GB and AB are each equal, by hipothesis, to DE, they are equal to each other; therefore the triangle ABG is isosceles, wherefore the angle BAG is equal to BGA (5. 1.).

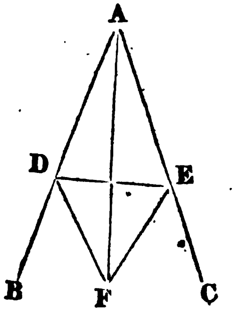
In the same manner it may be shown that the side AC is equal to CG, and the angle CAG to CGA. Therefore the two angles BAG, CAG together are equal to the two angles BGA, CGA together; that is, the whole angle BAC is equal to the whole angle BGC. But the angle BGC is, by hypothesis, equal to the angle EDF; therefore also the angle BAC is equal to EDF. Hence the triangles ABC, DEF are equal in all respects (4. 1.). Wherefore, if two triangles &c. Q. E. D.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle; that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle; it is required to bisect it.

Take any point D in AB, and from AC cut off AE equal to AD (3. 1.); join DE, and on it describe an equilateral triangle DEF (1. 1.); join AF; then the straight line AF bisects the angle BAC.



Because AD is equal to AE, and AF is common to the two triangles DAF, EAF, and DF is equal to EF; the angle DAF is equal to EAF (8. 1.); wherefore the angle BAC is bisected by the straight line AF. Which was to be done.

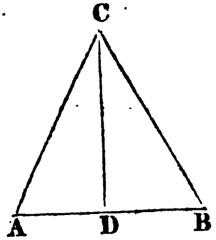
Scholium. In the same manner may each of the angles BAF, CAF be bisected. Therefore by successive subdivisions an angle may be divided into four, eight, sixteen, &c. equal parts. ED.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line; that is, to divide it into two equal parts.

Let AB be the given straight line; it is required to divide it into two equal parts.

On AB describe an equilateral triangle ABC (1. 1.), and bisect the angle ACB by the straight line CD (9. 1.); AB is bisected in the point D.



Because AC is equal to CB, and CD is common to the two triangles ACD, BCD, and the angle ACD is equal to BCD; the base AD is equal to DB (4. 1.); therefore the straight line AB is divided into two equal parts in the point D.

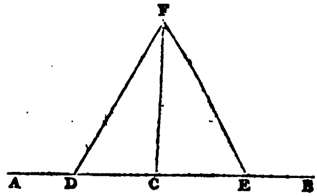
Cor. If a straight line bisect the vertical angle of an equilateral or an isosceles triangle, it will also bisect the base at right angles. For the triangles ACD, BCD are equal in all respects.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a point given in AB; it is required to draw a straight line from C at right angles to AB.

Take any point D in AC, and make CE equal to CD (3. 1.); on DE describe the equilateral triangle DFE (1. 1.), and join FC; the straight line FC, drawn from the given point C, is at right angles to the given straight line AB.



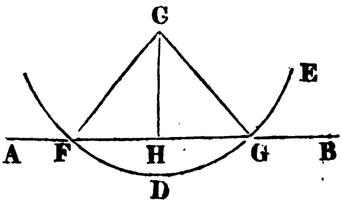
Because DC is equal to CE, and FC is common to the two triangles DCF, ECF, and the side DF is equal to EF; the angle DCF is equal to ECF (8. 1.); and they are adjacent angles; therefore each of the angles DCF, ECF, is a right angle (10 Def.). Wherefore from the given point C, in the given straight line AB, FC has been drawn at right angles to AB.

Note. If the given point C be at or near the end of the given line AB, then AB must be produced on the side of the point C. Ed.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without the line.

Let AB be a given straight line, which may be produced to any length both ways, and let C be a given point without the line; it is required to draw from C a straight line perpendicular to AB.



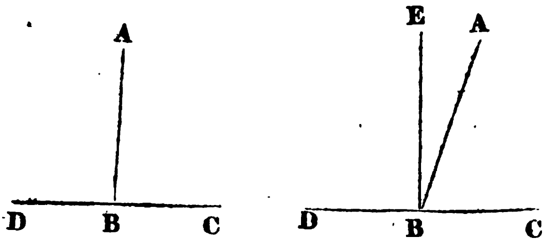
Take any point D on the other side of AB, and from the center C, with the radius CD, describe the circle EGF meeting AB in F and G (3 Post.); bisect FG in H (10. 1.), and draw CF, CH, CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.

Because FH is equal to HG, and HC is common to the two triangles FHC, GHC, and the side CF is equal to CG (20 Def.); the angle CHF is equal to CHG (8. 1.); and they are adjacent angles; therefore each of them is a right angle, and CH is perpendicular to FG (10 Def.). Therefore from the given point C a perpendicular CH has been drawn to the given straight line AB.

PROPOSITION XIII. THEOREM.

The two angles which one straight line makes with another, on one side of it, are together equal to two right angles.

Let the straight line AB make with CD, on one side of CD, the angles ABC, ABD; these are together equal to two right angles.



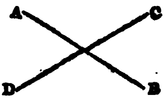
If the angle ABC be equal to ABD, each of them is a right angle (Def. 10.). But, if not, from the point B draw BE at right angles to CD (11. 1.); then the angles CBE, EBD are two right angles. The angle DBA is equal to the two angles DBE, EBA. To these equals add the angle CBA, then the two angles DBA, CBA will be equal to the three angles DBE, EBA, ABC. Now DBE is a right angle, and the two angels EBA, ABC are together equal to the right angle CBE. Therefore the sum of the three angles DBE, EBA, ABC is equal to two right angles. Consequently the sum of the two angles DBA, CBA is also equal to two right angles. Wherefore, the angles &c. Q. E. D. LEGENDRE.

Otherwise. The construction remaining, the angles CBE, EBD are right angles. But the angle ABD is greater than EBD by the angle ABE, and the angle ABC is less than EBC by the same angle ABE. Therefore the angles ABD, ABC are together equal to the angles CBE, EBD, that is, to two right angles. Wherefore, the angles &c. Q. E. D. ED.

Cor. 1. All the angles which any number of straight lines make with another straight line at the same point, and on the same side of the line, are together equal to two right angles.

For their sum is equal to the sum of the two angles DBA, CBA. ED.

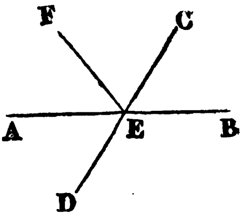
Cor. 2. If two straight lines cut each other, the four angles which they make at the point of intersection are together equal to four right angles.



For the two angles on each side of AB are together equal to two right angles, therefore the four angles on both sides of AB are together equal to four right angles. ED.

Cor. 3. All the angles made by any number of straight lines, which intersect or meet one another in one point, are together equal to four right angles.

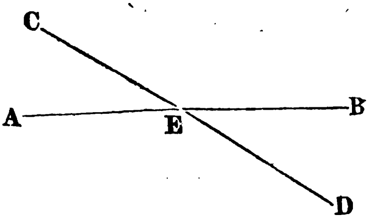
For all the angles made on one side of a straight line AB passing through the point of concourse E of any number of straight lines, are together equal to two right angles, by Cor. 1; therefore all the angles made at the point of concourse of the lines on both sides of AB are together equal to four right angles. Ed



PROPOSITION XV. THEOREM.

If two straight lines cut each other, the opposite angles will be equal.

Let the two straight lines AB, CD cut each other in the point E; the angle AEC will be equal to DEB, and the angle CEB to AED. For the angles CEA, AED which the straight line AE makes with CD, are together equal to two right angles (13. 1.); and the angles AED, DEB, which the straight line DE makes with AB, are together equal to two right angles;



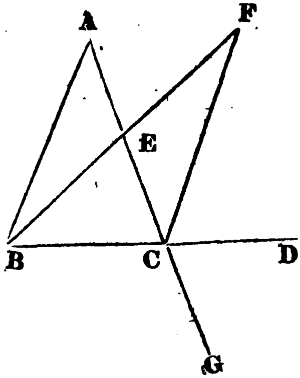
therefore the two angles CEA, AED are together equal to the two angles AED, DEB. Take away the common angle AED, and the remaining angles CEA, DEB are equal (3 Ax.). In the same manner it can be demonstrated that the angles CEB, AED are equal. Therefore, if two straight lines &c. Q. E. D.

Otherwise. This proposition is obvious, for the straight lines AB, CD have the same inclination to each other on both sides of the point of intersection E; therefore the opposite angles AEC, BED are equal, and also the opposite angles CEB, AED are equal. Q. E. D. Ed.

PROPOSITION XVI. THEOREM.*

If one side of a triangle be produced, the exterior angle will be greater than either of the interior and opposite angles.

Let ABC be a triangle and let one side BC be produced to D; the exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.



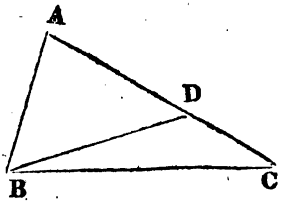
Bisect AC in E (10. 1.); join BE, and produce it to F; make EF equal to BE, and join FC.

Because AE is equal to EC, and BE to EF, the two sides AE, EB are equal to CE, EF, each to each; and the angle AEB is equal to the angle CEF (15. 1.); therefore the base AB is equal to the base CF (4. 1.), and the angle BAE is equal to the angle ECF. But the angle ECD is greater than the angle ECF; therefore the angle ECD, that is, ACD, is greater than BAC.

If the side AC be produced to G, and BC be bisected, it may be demonstrated in the same manner, that the angle BCG is greater than the angle ABC. But the angle BCG is equal to the vertical angle ACD (15. 1.). Therefore the angle ACD is greater than the angle ABC. Therefore, if one side &c. Q. E. D.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle subtends the greater angle, or has the greater angle opposite to it.



Let ABC be a triangle, of which the side AC is greater than AB; the angle ABC is also greater than BCA.

From AC cut off AD equal to AB and join BD. The exterior angle ADB of the triangle BDC is greater than the interior and opposite

* This proposition serves to demonstrate propositions 17, 18, and 27; but is otherwise useless.

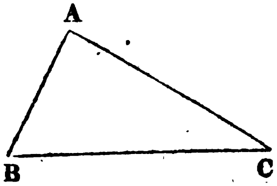
angle DCB (16. 1.). But ADB is equal to ABD (5. 1.), because the side AB is equal to AD; therefore the angle ABD is also greater than ACB; wherefore much more is the angle ABC greater than ACB. Therefore, the greater side &c. Q. E. D.

Cor. Hence, by an obvious and necessary consequence, the greater angle of every triangle is subtended by the greater side. Ed.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle B is greater than C; the side AC is also greater than AB.

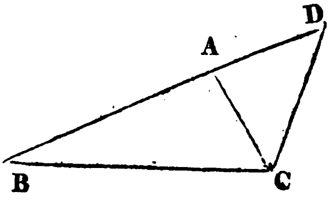


For If AC be not greater than AB, it must either be equal to or less than AB. But AC is not equal to AB, because then the angle B would be equal to C (5. 1.); but it is not; therefore AC is not equal to AB. Nor is AC less than AB, because then the angle B would be less than C (18. 1.); but it is not; therefore AC is not less than AB. And it has been shown that AC is not equal to AB; therefore AC is greater than AB. Wherefore, the greater angle &c. Q. E. D.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle; any two sides of it are together greater than the third side, namely, the sides BA, AC are greater than the side BC; and AB, BC greater than AC; and BC, CA greater than AB.



Produce BA to the point D, and make AD equal to AC (3. 1.), and join DC.

Because DA is equal to AC, the angle D is equal to ACD (5. 1.). But the angle BCD is greater than ACD; therefore BCD is greater than ADC; therefore the side DB is great-

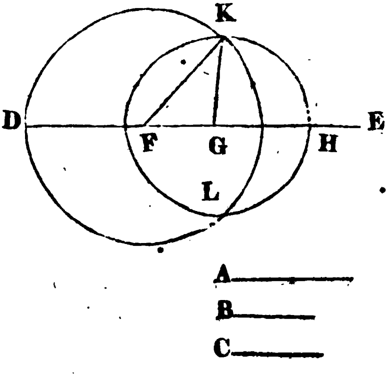
er than BC (19. 1.). But DB is equal to BA and AC together; therefore BA and AC together are greater than BC. In the same manner it may be proved that the sides AB, BC are together greater than AC; and BC, CA greater than AB. Therefore, any two sides &c. Q. E. D.

Otherwise. Because a straight line is the least distance between two points (4 Def.), the straight line BC is less than the two straight lines BA, AC together, or BA and AC are together greater than BC. Ed.

PROPOSITION XXII. PROBLEM.

To construct a triangle of which the sides shall be equal to three given straight lines; but any two of these lines must be greater than the third (20. 1.).

Let A, B, C be the three given straight lines, of which any two are greater than the third, namely, A and B greater than C; A and C greater than B; and B and C greater than A. It is required to make a triangle, of which the sides shall be equal to A, B, C, each to each.



Take a straight line DE unlimited toward D and E, and let F be any point in it; make DF equal to A (3. 1.), FG to B, and GH to C. Therefore any two of the three straight lines DF, FG, GH are together greater than the third. From the centre F at the distance FD, describe the circle DKL; and from the centre G at the distance GH, describe another circle HLK.

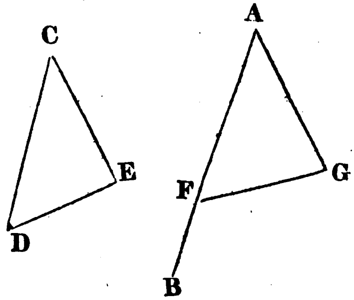
The circle DLK will cut FH, because FD is less than FG and GH, that is, than FH; and the circle HLK will cut GD, because GH is less than GF and FD, that is, than GD. The two circles DLK, HLK will cut each other, because their radii DF and GH are together greater than FG the distance between their centres. Let them intersect each other in K, and let KF, KG be joined. The triangle KFG has its sides equal to the three straight lines A, B, C.

Because the point F is the centre of the circle DKL, FD is equal to FK (20 Def.). But FD is equal to A; therefore FK is equal to A. Again because G is the centre of the circle LKH, GH is equal to GK. But GH is equal to C; therefore GK is equal to C. Now FG is equal to B. Therefore the three lines KF, FG, GK are equal to the three lines A, B, C, each to each; therefore the triangle KFG has its three sides KF, FG, GK equal to the three given lines A, B, C. Which was to be done.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the point A, in the line AB, which shall be equal to the angle C.



In CD, CE take any points D, E, and join DE; make the triangle AFG, the sides of which shall be equal to the three lines CD, DE, CE (Prop. 22.), so that CD be equal to AF CE to AG, and DE to FG. Because DC, CE are equal to FA, AG, each to each, and the base DE to FG, the angle C is equal to A (Prop. 8). Therefore at the given point A, in the given straight line AB, the angle FAG is made equal to the given rectilineal angle C.

Scholium. If the parts CD, CE be taken equal to each other, the construction will be more commodious. ED.

PROPOSITION XXVI. THEOREM.

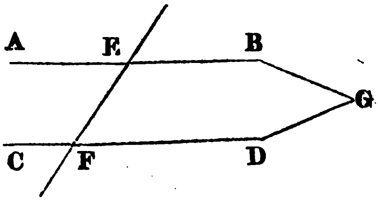
This is Prop. A, page 25.

PROPOSITION XXVII. THEOREM.

If a straight line intersect two other straight lines which are in the same plane, and make the alternate angles equal to each other, those two lines are parallel. *See Appendix to Book I.*

Let the straight line EF, which intersects the two straight lines AB, CD, make the alternate angles AEF, EFD equal to each other; AB is parallel to CD.

For if AB and CD be not parallel, they are inclined to each other, and, if produced, will meet either toward B, D, or toward A, C. Let them be produced and meet toward B, D, in the point G; then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG (Prop. 16.). But the angle AEF is also equal to EFG, which is impossible; therefore AB and CD being produced do not meet toward B, D. In like manner it may be demonstrated that they do not meet toward A, C. Therefore AB, CD are not inclined to each other. Therefore AB is parallel to CD (14 Def.). Wherefore, if a straight line &c. Q. E. D.*

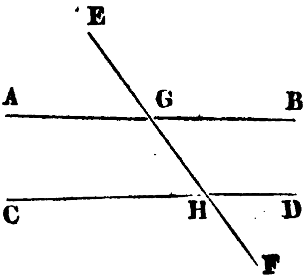


PROPOSITION XXVIII. THEOREM.

If a straight line intersect two other straight lines which are in the same plane, and make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the two interior angles on the same side together equal to two right angles; the two straight lines are parallel.

Let the straight line EF, which intersects the two straight lines AB, CD, make the exterior angle EGB equal to the inte-

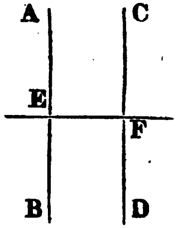
* By reason of the false supposition, and construction of the figure, this demonstration is not intelligible to learners.



rior and opposite angle GHD on the same side; or make the interior angles BGH, GHD on the same side, together equal to two right angles; AB is parallel to CD.

Because the angle EGB is equal to GHD, and EGB is also equal to the angle AGH (Prop. 15.), the angle AGH is equal to GHD; and they are alternate angles; therefore AB is parallel to CD (Prop. 27.).

Again, the angles BGH, GHD are together equal to two right angles (by Hyp.), and AGH, BGH, are together equal to two right angles (Prop. 13.); therefore the angles AGH, BGH, are together equal to the angles BGH, GHD. Take away the common angle BGH, then the remaining angles AGH, GHD are equal; and they are alternate angles; therefore AB is parallel to CD. Wherefore, if a straight line &c. Q. E. D.



Cor. Straight lines which are perpendicular to the same straight line, and in the same plane with it, are parallel to one another.

If AB and CD be perp. to EF, the angles at E and F will be right angles (10 Def.), and therefore equal (11 Def. Cor.); wherefore AB is parallel to CD. Ed.

PROPOSITION XXIX. THEOREM.

If a straight line intersect two parallel straight lines it makes the alternate angles equal to each other; and the exterior angle equal to the interior and opposite angle on the same side of it; and likewise the two interior angles on the same side together equal to two right angles.

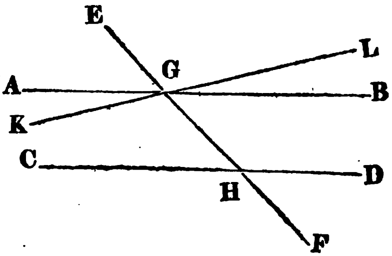
Axiom 10. *Two straight lines which intersect each other cannot be both parallel to the same straight line.**

Let the straight line EF intersect the parallel straight lines AB, CD; the alternate angles AGH, GHD are equal to each

* This axiom was omitted in its proper place by mistake.

other; and the exterior angle EGB is equal to the interior and opposite angle GHD on the same side of EF; and the two interior angles BGH, GHD, on the same side, are together equal to two right angles.

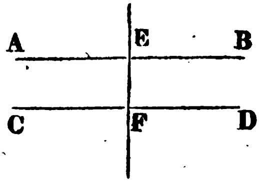
If AGH be not equal to GHD, one of them must be greater than the other. Let AGH be the greater, and at the point G, in the straight line EF, make the angle KGH equal to GHD (Prop. 23.), and produce KG to L; KL will be parallel to CD (Prop. 27). But AB is also parallel to CD; therefore two straight lines AB, KL, which intersect each other, are parallel to the same line CD, which is impossible (10 Ax.). Therefore the angles AGH, GHD are not unequal.



Again, the angle EGB is equal to AGH (Prop. 15.); and AGH is proved to be equal to GHD; therefore the angle EGB is likewise equal to GHD. Add to each of these the angle BGH, then the angles EGB, BGH are together equal to the angles BGH, GHD. But EGB, BGH are together equal to two right angles (Prop. 13); therefore also BGH, GHD are together equal to two right angles. Wherefore, if a straight line &c. Q. E. D.

Cor. If a straight line EF be perpendicular to one CD of two parallel straight lines AB, CD, it is also perpendicular to the other AB.

For the angle AEF is equal to EFD, and the angle BEF is equal to EFC. But the angles EFD and EFC are right angles, because EF is perp. to CD. Therefore AEF and BEF are right angles, therefore EF is perpendicular to AB. Ed.

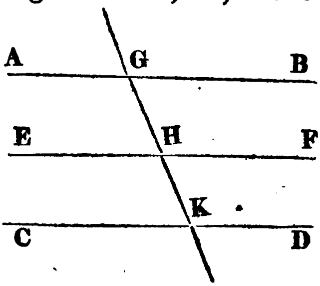


PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

Let the straight lines AB, CD be parallel to EF; AB is also parallel to CD.

Let the line GHK cut the straight lines AB, EF, CD. Because GHK cuts the parallel lines AB, EF, the angle AGH is equal to GHF (Prop. 29); and because GK cuts the parallel lines EF, CD, the angle GHF is equal to GKD; therefore also the angle AGK is equal to GKD; and they are alternate angles; therefore AB is parallel to CD (Prop. 27). Wherefore straight lines &c. Q. E. D.

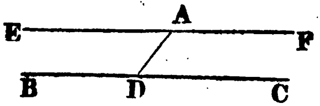


PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line through A parallel to BC.

In BC take any point D, and join AD at the point A, in the line AD, make the angle DAE equal to ADC (Prop. 23); and produce the line EA to F. Then EF is parallel to BC.



Because AD meets the two straight lines BC, EF, and makes the alternate angles EAD, ADC equal to each other, EF is parallel to BC (Prop. 27). Therefore the line EAF is drawn through the given point A parallel to the given line BC. Which was to be done.

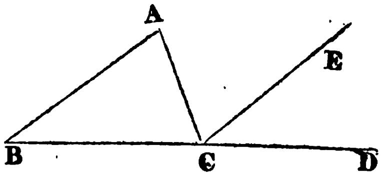
PROPOSITION XXXII. THEOREM.*

1. If any side of a rectilineal triangle be produced, the exterior angle is equal to the two interior and opposite angles taken together.

Let ABC be a triangle, and let a side BC be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC together.

*Euclid's enunciation contains two distinct propositions, which are enunciated and demonstrated separately, as follows.

Through the point C draw CE parallel to the side AB (Prop. 31.). Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (Prop. 29.); and because AB is parallel to



CE, and BD meets them, the exterior angle ECD is equal to the interior and opposite angle B. Hence the whole exterior angle ACD is equal to the two interior and opposite angles A and B together. Therefore, if any side &c. Q. E. D.

Cor. The difference between the exterior angle and either of the interior and opposite angles is equal to the other interior and opposite angle.

2. The three angles of every rectilineal triangle are together equal to two right angles.

Let ABC be a triangle; the three angles A, B, C, are together equal to two right angles.

Produce any side BC to D, then the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, together. To these equals add the angle ACB, then the two angles ACD, ACB are together equal to the three angles CBA, BAC, ACB. But the angles ACD, ACB are together equal to two right angles (Prop. 13.); therefore the three angles CBA, BAC, ACB are together equal to two right angles. Wherefore the three angles &c. Q. E. D.

Cor. 1. If one angle in one triangle be equal to one angle in another, the other two angles of the former triangle are together equal to the other two angles of the latter. Ed.

Cor. 2. If two angles of one triangle be together equal to two angles of another, the third angle of the former triangle is equal to the third angle of the latter (3 Ax.). Ed.

Cor. 3. If one angle in any triangle be either right or obtuse, then each of the other two angles is acute. Ed.

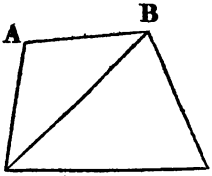
Cor. 4. If one angle in any triangle be a right angle, then the other two angles together are equal to a right angle. Ed.

Note. Two such angles are called *complements* of each other to a right angle.

Cor. 5. Each angle of an equilateral triangle is one third of two right angles, or two thirds of one right angle.

Let A denote each angle, then $3A = 2$ right angles, therefore $A = \frac{1}{3}$ of 2 right angles, or $\frac{2}{3}$ of one right angle. Ed.

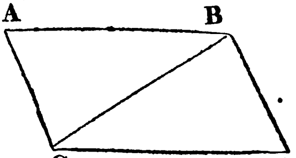
Cor. 6. In every quadrilateral figure the sum of the four angles is equal to four right angles.



For if a line BC be drawn between two opposite angles of any quadrilateral figure ACDB, it will divide the figure into two triangles ABC, BCD. Now the sum of the three angles of each triangle is equal to two right angles; therefore all the angles of both triangles, which make the four angles of the quadrilateral figure, are together equal to four right angles. Ed.

PROPOSITION XXXIII. THEOREM.

The two straight lines which join the corresponding extremities of two equal and parallel straight lines are also equal and parallel.



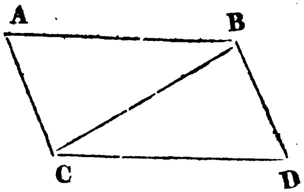
Let AB, CD be two equal and parallel lines, and let the lines AC, BD join their corresponding extremities; then AC, BD are also equal and parallel.

Draw the diagonal BC. Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (Prop. 29). Because the side AB is equal to CD, and BC is common to the two triangles ABC, DCB, and the angle ABC is equal to BCD, the base AC is equal to BD (Prop. 4.), and the angle ACB to CBD. Because the line BC meets the two lines AC, BD, and makes the alternate angles ACB, CBD equal to each other, AC is parallel to BD (Prop. 27.); and AC was shown to be equal to BD. Therefore, the two straight lines &c. Q. E. D.

PROPOSITION XXXIV. THEOREM.

The two opposite sides of a parallelogram are equal to each other, and also the two opposite angles; and the diagonal of a parallelogram divides it into two equal triangles.

Let ACDB be a parallelogram, of which BC is a diagonal; the opposite sides of the figure are equal to each other, and also the opposite angles; and the diagonal BC bisects it.



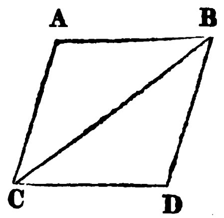
Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to each other (Prop. 29); and because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to each other; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one equal to two agles BCD, CBD in the other, each to each, therefore the remaining angles BAC, BDC are equal (2 Cor. 32.), and the whole angles ABD, ACD are equal. Hence the opposite angles of a paral. are equal.

Again, the triangles ABC, BCD are mutually equiangular, and the side BC is common to both, therefore they are equal in all respects (A), and the side AB is equal to CD, and the side AC to BD, and the triangle ABC to BCD. Therefore the opposite sides of a paral. are equal, and the diagonal bisects it. Wherefore the opposite sides &c. Q. E. D.

PROPOSITION B. THEOREM. ED.

If the opposite sides of a quadrilateral figure be equal, they are also parallel, that is, the figure is a parallelogram.

Let ABDC be a quadrilateral figure, having its opposite sides equal, namely, AB equal to CD, and AC equal to BD; the opposite sides are parallel.



Draw the diagonal BC; then, because the side AB is equal to CD, and the side AC to BD, and the side BC is common to the triangles ACB, BCD, the angles ABC, BCD are equal (Prop. 8.) and the angles ACB, CBD are equal. But ABC, BCD are alternate angles; there-

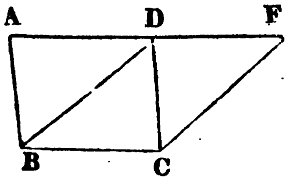
fore AB, CD are parallel (Prop. 27.). Also ACB, CBD are alternate angles; therefore AC, BD are parallel. Hence ABDC is a parallelogram. Therefore, if the opposite sides &c. Q. E. D.

Cor. If the opposite sides of a quadrilateral figure be equal, the opposite angles are also equal. For the figure is a parallelogram, therefore the opposite angles are equal.

PROPOSITION XXXV. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal to one another; that is, their surfaces are equal.

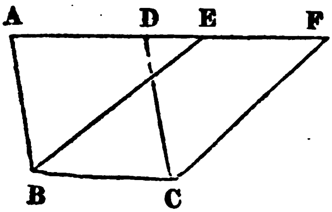
Let the parallelograms ABCD, DBCF be on the same base BC, and between the same parallels AF, BC; they are equal in surface.

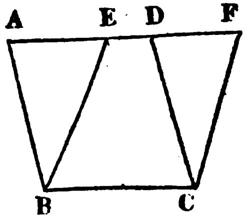


If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain that each of the parallelograms is double of the triangle BDC (Prop. 34); therefore they are equal to each other.

But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point; then, because ABCD is a paral. AD is equal to BC (Prop. 34). For the same reason EF is equal to BC. Wherefore AD is equal to EF. From the whole line AF take the two equal parts AD, EF, and the remainders DF, AE will be equal.

Now AB is equal to DC; and the exterior angle FDC is equal to the interior angle EAB (Prop. 29); wherefore the tri-



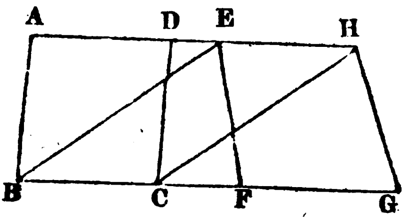


angle EAB is equal to the triangle FDC (Prop. 4). Take the triangle FDC from the trapezium ABCF, and the paral. ABCD

will remain; and take the triangle AEB from the trap. ABCF, and the paral. EBCF will remain. Therefore the paral. ABCD is equal to EBCF. Therefore, parallelograms &c. Q. E. D.

PROPOSITION XXXVI. THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal to one another.



Let ABCD, EFGH be parals. on equal bases BC, FG, and between the same parallels AH, BG; the paral. ABCD is equal to EFGH.

Join BE, CH. Because BC is equal to FG, and FG to EH (Prop. 34), BC is equal to EH. But BC, EH are parallels, and joined at their corresponding extremities by the lines BE, CH; therefore BE, CH are equal and parallel (Prop. 33); therefore EBCH is a paral. and it is equal to ABCD (Prop. 35). But the paral. EFGH is equal to EBCH; therefore also the paral. ABCD is equal to EFGH. Wherefore, parallelograms &c. Q. E. D.

Otherwise. This is an obvious deduction from the last proposition, and needs no formal proof by a figure.

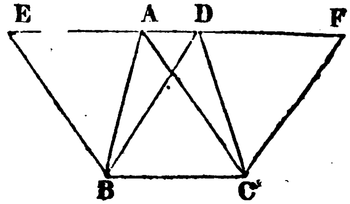
Suppose the base of one paral. to be applied to the base of the other; then the bases will coincide, because they are equal, and the two parals. will stand on the same base and between the same parallels. Therefore they are equal to each other. Therefore, parallelograms &c. Q. E. D. Ed.

PROPOSITION XXXVII. THEOREM.

Triangles on the same base, and between the same parallels, are equal to one another; that is, their surfaces are equal.

Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC; the triangle ABC is equal to DBC.

Produce AD both ways to the points E, F; through B draw BE parallel to CA (Prop. 31), and through C draw CF parallel to BD; then each of the figures EBCA, DBCF is a paral.; and EBCA is equal to DBCF (Prop. 35).



Now the triangle ABC is half of the paral. EBCA (Prop. 34); and the triangle DBC is half of the paral. DBCF; therefore the triangle ABC is equal to DBC (7 Ax.). Wherefore, triangles &c. Q. E. D.

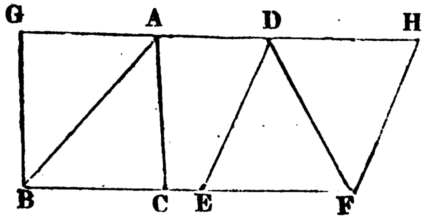
Otherwise. This is a cor. to prop. 35. For triangles are the halves of parallelograms on the same base and between the same parallels (Prop. 34.), and therefore are equal to one another. ED.

PROPOSITION XXXVIII. THEOREM.

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD; the triangle ABC is equal to DEF.

Produce AD both ways to the points G, H; through B draw BG parallel to CA (Prop. 31), and through F draw FH parallel



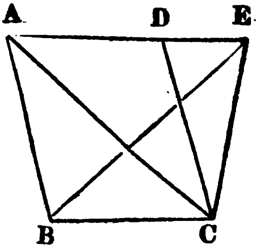
to ED; then the figures GBCA, DEFH are parallelograms; and they are equal (Prop. 36).

Now the triangle ABC is half of the paral. GBCA (Prop. 34), and the triangle DEF is half of the paral. BEFH; therefore the triangle ABC is equal to DEF (7 Ax.). Wherefore, triangles &c. Q. E. D.

Otherwise. This is a cor. to prop. 36. For they are the halves of parals. on equal bases and between the same parallels, and therefore are equal. ED.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be on the same base, and between the same parallels, the parallelogram is double of the triangle.



Let the paral. ABCD and the triangle EBC be on the same base BC, and between the same parallels BC, AE; the paral. ABCD is double of the triangle EBC.

Join AC; then the triangle ABC is equal to EBC (Prop. 37). But the paral. ABCD is double of the triangle ABC (Prop. 34); wherefore ABCD is also double of the triangle EBC. Therefore, if a parallelogram &c. Q. E. D.

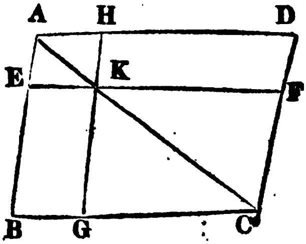
Cor. If a parallelogram and a triangle be between the same parallels, and if the base of the paral. be half the base of the triangle, the paral. will be equal to the triangle.

For if a paral. were described on half the base BC, and between the parallels BC, AE, it is evident that it would be equal to half the paral. ABCD, and therefore equal to the triangle EBC on double the base. Ed.

PROPOSITION XLIII. THEOREM.

The complements of the two parallelograms which are about the diagonal of any parallelogram are equal to each other.

Let ABCD be a paral. of which the diagonal is AC; let EH, FG be the parals. about AC, that is, through which AC passes; and let BK, KD be the other parals. which complete the whole figure ABCD, and are therefore called the *complements*. The complements BK, KD are equal.



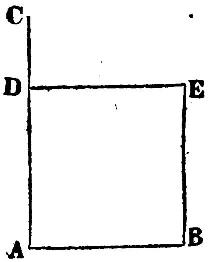
Because ABCD, AEKH, KGCF are parallelograms, and AC is their common diagonal, the triangle ABC is equal to ADC (Prop. 34), and the triangle AEK to AHK, and the triangle KGC to KFC; therefore the complements BK, KD are equal (3 Ax.). Therefore, the complements &c. Q. E. D. Ed.

PROPOSITION XLVI. PROBLEM.

To describe a square on a given straight line.

Let AB be the given straight line on which it is required to describe a square.

From the point A draw AC perp. to AB, and make AD equal to AB; through D draw DE parallel to AB (Prop. 31), and through B draw BE parallel to AD; then ADEB is a square described on AB. For ADEB is a paral., therefore AB is equal to DE (Prop. 34), and AD to BE. But BA is equal to AD; therefore the four lines BA, AD, DE, EB, are equal to one another, and the paral. ADEB is equilateral.



The line AD meeting the parallels AB, DE, makes the angles BAD, ADE equal to two right angles (Prop. 29). But BAD is a right angle; therefore ADE is a right angle. Now the angle B is equal to its opposite angle D, and the angle E to A (Prop. 34); therefore each of the angles B, E is a right angle; wherefore the figure ADEB is rectangular. But it is equilateral; therefore it is a square; and it is described on the given straight line AB. Which was to be done.

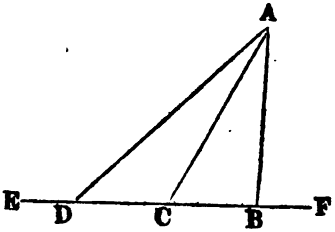
Cor. A parallelogram having one angle right, and two adjacent sides equal, is a square.

PROPOSITION XLVII. THEOREM.

See Prop. B, Book II.

PROPOSITION C. THEOREM. Ed.

If straight lines be drawn from the same point to meet a given straight line, the perpendicular is the least, and any line which is nearer the perpendicular than another is less than the other.



Let A be any point, and EF a given straight line; and let AB, AC, AD be straight lines drawn from A to meet EF.

Let AB be perp. to EF; AB is less than AC, and AC is less than AD. For since B is a right angle, ACB is an acute angle (3 Cor. 32), therefore the side AC is greater than AB (Prop. 19). Again, because ACB is an acute angle, ACD is obtuse (Prop. 13), therefore the angle ADC is acute, therefore the side AD is greater than AC, and still greater than AB. Therefore, if straight lines &c. Q. E. D.

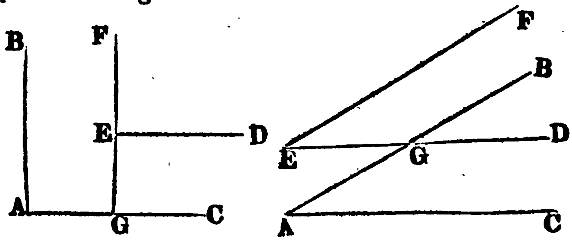
Cor. The distance between a point and a straight line is the perpendicular drawn from the point to the line.

Scholium. The term *distance* signifies the shortest interval between two points; therefore the distance between two points is the straight line which joins them.

PROPOSITION D. THEOREM. Ed.

If the sides of one angle be parallel to the sides of another angle, each to each, and be directed the same way, the angles are equal to each other.

Let the sides AC, AB of the angle BAC be parallel to the sides ED, EF of the angle DEF, each to each; the angle BAC is equal to the angle DEF.



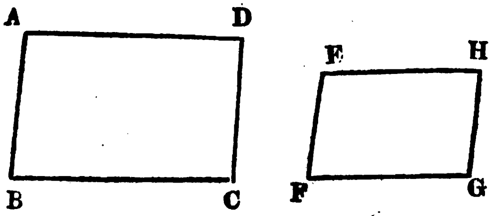
Produce FE to meet AC in G (in the first figure), then, because BA is parallel to FG, the angle BAC is equal to FGC; and because AC is parallel to ED, the angle FED is equal to FGC; therefore the angle BAC is equal to FED.

Again (in the second figure), because AC is parallel to ED, the angle BAC is equal to the alternate angle AGE (Prop. 29); and because AB is parallel to EF, the angle FED is equal to AGE; therefore the angle BAC is equal to FED. Therefore, if the sides &c. Q. E. D.

PROPOSITION E. THEOREM. Ed.

If two parallelograms have one angle in one parallelogram equal to one angle in the other, the remaining angles of the former are equal to the remaining corresponding angles of the latter.

Let ABCD, EFGH be two parallelograms, which have the angles at A and E equal; the other angles of the former paral. are equal to the other angles of the latter, each to each, namely, the angle B is equal to F, the angle C to G, and the angle D to H.

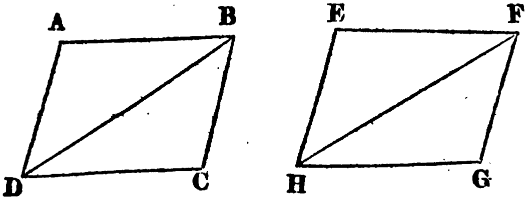


For in the paral. ABCD the two angles A and B are together equal to two right angles (Prop. 29); and in the paral. EFGH the two angles E and F are together equal to two right angles. Therefore the angles A and B are together equal to the angles E and F. Therefore if from these equals the equal angles A and E be taken, the remaining angles B and F will be equal.

Again, in the paral. ABCD the opposite angles A and C are equal (Prop. 34), and in the paral. EFGH the opposite angles E and G are equal. But the angles A and E are equal, therefore the angles C and G are equal. In the same manner it may be proved that the angle D, to which B is opposite, is equal to the angle H, to which F is opposite. Therefore, if two parallelograms &c. Q. E. D.

PROPOSITION F. THEOREM. Ed.

If two parallelograms have two adjacent sides of one equal to two adjacent sides of the other, each to each; and have also the angles contained by those sides equal to each other; the parallelograms will be equal in all respects.



Let ABCD, EFGH be two parallelograms, which have the two sides AB, AD equal to the two sides EF, EH, each to each, and the angle A equal to E; the parallelograms are equal, and the other corresponding sides are equal, and the other corresponding angles are equal.

Draw the diagonals BD, FH, then the triangle ABD is equal to BCD (Prop. 34), and the triangle EFH is equal to FGH. But the triangle ABD is equal to EFH (Prop. 4), therefore the triangle BCD is equal to FGH. Consequently the paral. ABCD is equal to EFGH.

Again, the side AB is equal to DC, and the side AD is equal to BC (Prop. 34); also the side EF is equal to HG, and the side EH is equal to FG. But the side AB is equal to EF, and the side AD is equal to EH. Consequently the side DC is equal to HG, and the side BC is equal to FG.

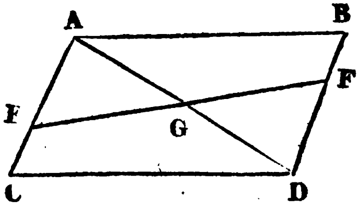
Lastly, the angle A is equal to C, and the angle E is equal to G. But the angle A is equal to E, therefore the angle C is equal to G. Consequently the two angles ABC, ADC are together equal to the two angles EFG, EHG together (6 Cor. 32). Hence those four angles are equal to one another. Therefore, if two parallelograms &c. Q. E. D.

Cor. 1. Two rectangles contained by equal straight lines are equal to each other.

Cor. 2. Squares which stand on equal straight lines are equal to one another.

PROPOSITION G. THEOREM. Ed.

A straight line passing through the middle of the diagonal of a parallelogram divides the parallelogram into two equal parts.



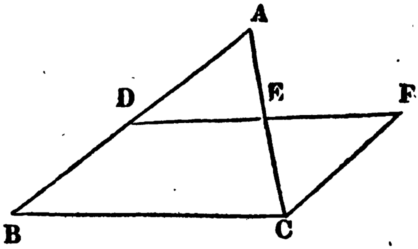
Let ABDC be a paral., AD a diagonal, and EF a line passing through its middle G; the figure ABFE is equal to EFDC.

For the triangles ABD, ACD are equal, and the triangles AEG, DGF are equiangular, and have the sides AG, DG equal, therefore they are equal (Prop. A). Conseq. if they be taken from the triangles ABD, ACD, the remainders ABFG, CDGE will be equal. Hence it is evident that ABFG and AGE are together equal to CDGE and DGF together, or ABFE, CDFE. Therefore, a straight line &c. Q. E. D.

Cor. Any straight line EF drawn through the middle of the diagonal of a parallelogram is bisected by the diagonal.

PROPOSITION H. THEOREM. Ed.

If two sides of a triangle be bisected, the straight line which joins the points of section will be parallel to the third side, and equal to half of it.



Let the sides AB, AC of the triangle ABC be bisected in the points D, E; the straight line DE joining those points will be parallel to BC, and equal to half of BC.

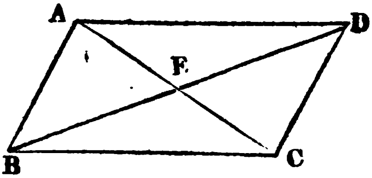
In DE produced take EF equal to DE, and join CF. Because the side DE is equal to EF, and AE equal to EC, and the angle AED equal to CEF, the triangles ADE, FEC are equal in all respects, therefore the side AD is equal to CF, and the angle ADE equal to the alternate angle CFE, therefore AB is parallel to FC.

Because BD or AD and CF are equal and parallel, the straight lines BC, DF joining their extremes are also equal and parallel (Prop. 33). But DF is double of DE, therefore BC is double of DE. Therefore, if two sides &c. Q. E. D.

PROPOSITION I. THEOREM. Ed.

The diagonals of any parallelogram bisect each other, and divide the parallelogram into four equal triangles.

1. Let ABCD be any parallelogram, of which the diagonals are AC and BD; they will bisect each other.



Let AC, BD intersect each other in E; then, in the triangles ADE, BCE, the vertical angles at E are equal (Prop. 15), and the alternate angles EAD, ECB are equal (Prop. 29), and the alternate angles EDA, EBC are equal, and the sides AD, BC are equal (Prop. 34); therefore the other sides which are opposite to equal angles are also equal (Prop. A), namely, AE to EC, and DE to EB. Therefore the diagonals AC, BD bisect each other in E.

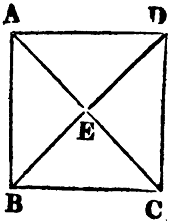
2. The diagonals AC, BD divide the parallelogram into four equal triangles.

For the triangles ABE, ADE being on equal bases BE, ED (by case 1), and between the same parallels, are equal; and the triangles ADE, CDE being on equal bases AE, EC, and between the same parallels, are equal; and the triangles CDE, CBE being on equal bases DE, EB, and between the same parallels, are equal. Consequently the four triangles ABE, ADE, CDE, CBE, are equal to one another. Therefore, the diagonals &c. Q. E. D.

PROPOSITION K. THEOREM. Ed.

The diagonals of a square bisect each other, and divide the square into four right angled triangles which are equal in all respects.

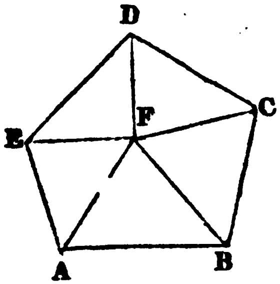
Let ABCD be a square, of which the diagonals are AC, BD; the four triangles ABE, ADE, CBE, CDE, are right angled, and are equal in all respects.

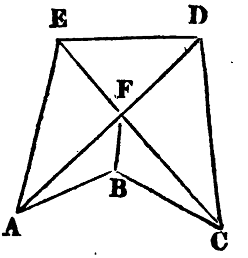


For the diagonals AC, BD bisect each other (Prop. I.), and divide the square into four triangles which are mutually equilateral, and therefore are equal in all respects, therefore the angles at E are right angled. Therefore, the diagonals &c. Q. E. D.

PROPOSITION L. THEOREM.

All the interior angles of any rectilineal figure are together equal to twice as many right angles as the figure has sides, wanting four right angles.





Let ABCDE be any rectilineal figure; all its interior angles A, B, C, D, E, are together equal to twice as many right angles, wanting four, as the figure has sides.

For any rectilineal figure ABCDE can be divided into as many triangles as it has sides, by drawing lines from a point F within the figure to each of its angles; and the three angles of each triangle are together equal to two right angles (Prop. 32), therefore all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure. But the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is, together with four right angles (3 Cor. 13). Therefore, twice as many right angles as the figure has sides are equal to all the angles of the figure, together with four right angles: that is, the angles of the figure are equal to twice as many right angles as the figure has sides, wanting four. Therefore, all the interior angles &c. Q. E. D.

Cor. All the interior angles of any quadrilateral figure are together equal to four right angles.

Scholium. Let n denote the number of the sides of a polygon, s the sum of all the interior angles, and r a right angle; then the proposition will be expressed by this theorem, $s=2r \times n-4r$, by means of which the sum of all the interior angles of any convex polygon may be found. Thus, if the figure be quadrilateral, then $n=4$, and $s=2r \times n-4r=8r-4r=4r$, that is, four right angles, which answers to 6 Cor. 32. If $n=5$, then $2r \times n-4r=10r-4r=6r$.

If the figure proposed be an equiangular polygon, or have all its angles equal, the quantity of each angle may be found by this formula. Suppose an equiangular polygon of six sides, then $2r \times n-4r=12r-4r=8r$, therefore $\frac{8}{6}r=\frac{4}{3}r=\frac{4}{3} \times 90=4 \times 30=120$, that is, each of the angles of a regular hexagon is 120 degrees. Let the figure be a regular octagon, then $2r \times n-4r=12r$, therefore $\frac{12}{8}r=\frac{3}{2}r=\frac{3}{2} \times 90=3 \times 45=135$ degrees.

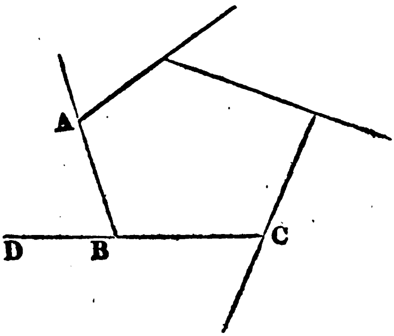
Let $s=2r \times n-4r=(2n-4) \times r=(n-2) \times 2r$, that is, all the interior angles of any convex rectilineal figure are together equal to twice as many right angles as the figure has sides less two sides. Ed.

PROPOSITION M. THEOREM.

All the exterior angles of any convex rectilineal figure, made by producing all the sides outward in the same direction, are together equal to four right angles.

Let A, B, C, &c. be the exterior angles of any rectilineal figure, made by producing its sides; the exterior angles A, B, C, &c. are together equal to four right angles.

Because every interior angle ABC of the figure, with its adjacent exterior angle ABD, are equal to two right angles (Prop. 13); and because there are as many exterior or interior angles as the figure has sides; all the interior together with all the exterior angles of the figure are equal to twice as many right angles as there are sides of the figure. But all the interior angles together with four right angles



are equal to twice as many right angles as there are sides of the figure (Prop. L). Therefore all the interior and all the exterior angles are together equal to all the interior angles together with four right angles. From these equals take away all the interior angles, and all the exterior angles will be equal to four right angles. Therefore, all the exterior &c. Q. E. D.

Scholium. The propositions in this Book are not arranged in the order of the subjects, but in such order as to enable the Author to demonstrate certain propositions by means of others which are placed before them. In the following abstract they are disposed according to the nature of the subjects, and such theorems as are of little or no use are omitted. Ed.

THE PRINCIPAL THEOREMS IN BOOK I.

Properties of Straight Lines and Angles:

If one straight line meet another, the sum of the two adjacent angles is equal to two right angles.

If any number of straight lines intersect or meet one another in the same point, the sum of all the angles about that point is equal to four right angles.

If two straight lines intersect each other, the vertical angles are equal.

Properties of Parallel Lines.

If a straight line intersect two straight lines, and make the alternate angles equal, or the exterior angle equal to the interior and opposite angle on the same side of the cutting line, or the sum of the two interior angles on the same side equal to two right angles, then those two lines are parallel.

A straight line intersecting two parallel straight lines makes the alternate angles equal, and the exterior angle equal to the interior and opposite angle on the same side of the cutting line, and the sum of the two interior angles on the same side equal to two right angles.

Straight lines which are parallel to the same straight line are parallel to one another.

Mutual Equality of Triangles.

When the three sides of one triangle are respectively equal to the three sides of another, the angles opposite to equal sides are also equal, and the triangles are equal.

When two triangles are mutually equiangular, and have two corresponding sides equal, the other corresponding sides (or the sides opposite to equal angles in each triangle) will be equal.

When two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, then the third sides are equal, and the angles opposite to equal sides are also equal.

Properties of Triangles.

In every triangle the greater of any two sides subtends the greater angle; and, conversely, the greater of any two angles is subtended by the greater side.

The sum of any two sides of a triangle is greater than the third side.

If one side of any triangle be produced, the external angle is equal to both the internal and opposite angles.

The sum of the three angles of every triangle is equal to two right angles.

If two angles of one triangle be together equal to two angles of another, the third angle of the former triangle is equal to the third angle of the latter.

If one angle of any triangle be a right angle, the sum of the other two angles is equal to a right angle.

If one angle of any triangle be either right or obtuse, each of the other two angles is acute.

Triangles on the same base, or on equal bases, and between the same parallels, are equal.

If two sides of a triangle be equal, the opposite angles are equal; and if two angles of a triangle be equal, the opposite sides are equal.

Properties of Parallelograms.

Straight lines which join the corresponding extremities of two equal and parallel straight lines are also equal and parallel; that is, those four lines make a parallelogram.

The opposite sides of a parallelogram are equal, and also the opposite angles.

The diagonal of a parallelogram divides it into two equal triangles.

If the opposite sides of a quadrilateral figure be equal, they are also parallel; and if the opposite angles of a quadrilateral figure be equal, the opposite sides are parallel; that is, in both cases the figure is a parallelogram.

The diagonals of any parallelogram bisect each other.

Parallelograms on the same base, or on equal bases, and between the same parallels, are equal.

If a parallelogram and a triangle stand on the same base and between the same parallels, the parallelogram is double of the triangle.

The complements of the two parallelograms which are about the diagonal of any parallelogram are equal to each other.

General Properties of Rectilinear Figures.

The sum of all the internal angles of any rectilinear figure is equal to twice as many right angles, except four, as the figure has sides.

If all the sides of any rectilinear figure be produced outward, the sum of all the external angles is equal to four right angles.

The sum of the four angles of every quadrilateral figure is equal to four right angles.

APPENDIX.

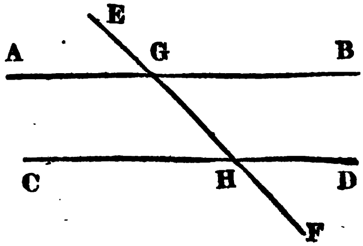
The demonstrations of the properties of parallel lines are seldom understood by learners. Indeed the subject of parallel lines is allowed to be one of the most difficult in the elements of geometry, and has exercised the ingenuity of many modern geometers, who have not been able to remove all the difficulties without impairing the rigour of geometrical demonstration. Some have attempted to give more simple demonstrations than those of Euclid, by means of new axioms, and others by means of a number of auxiliary propositions.

The following method of demonstrating the properties of parallel lines is plain and intelligible to youth, and appears to be legitimate. The definition of parallel lines which is here adopted has been proposed by some skilful mathematicians, and seems to be unexceptionable.

Definition. Parallel lines are such as lie in the same plane, and have no inclination to one another; that is, if produced to any length both ways they neither approach to nor recede from one another.

Cor. 1. Hence parallel lines are equidistant.

Cor. 2. Hence a straight line EF intersecting two parallel straight lines AB, CD, has the same inclination to both, that is, it makes the external angle EGB equal to the internal angle EHD on the same side of it.



If EF be perp. to one AB, it will be perp. to the other CD; if it cross one obliquely, it will also cross the other with the same obliquity.

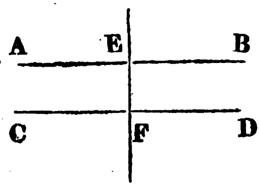
PROPOSITION I. THEOREM.

A straight line intersecting two parallel straight lines makes the alternate angles equal, and the two interior angles on the same side of it together equal to two right angles.—*See figure on next page.*

Let the straight line EF intersect the two parallel straight lines AB, CD; the alternate angles AGH, GHD are equal, and the angles BGH, GHC are equal; and the two interior angles BGH, GHD, on the same side of EF, are together equal to two right angles.

Since AB, CD are parallel, EF has the same inclination to each of them, or the angles EGB, EHD are equal (2 Cor. Def.); and because the two lines AB, EF intersect each other, the vertical angles AGH, EGB are equal (Prop. 15); consequently the angle AGH is equal to GHD. In like manner the angle BGH is equal to GHC.

Again, the angles BGE, BGH are together equal to two right angles (Prop. 13), and the angle BGE is equal to GHD; therefore the angles BGH, GHD are together equal to two right angles. Therefore a straight line &c. Q. E. D.



Cor. If a straight line EF be perpendicular to one AB of two parallel straight lines AB, CD, it is also perpendicular to the other CD.

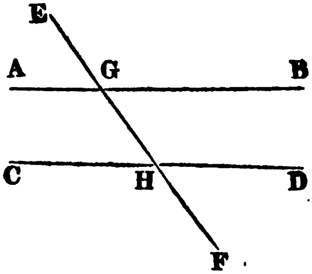
For the angle AEF is equal to the alternate angle EFD. But AEF is a right angle, therefore EFD is a right angle, therefore EF is perp. to CD.

PROPOSITION II. THEOREM.

If a straight line intersect two other straight lines, and make the alternate angles equal to each other, or make the two internal angles, on the same side of it,

together equal to two right angles; then those two lines are parallel to each other.

Let a straight line EF intersect two other straight lines AB, CD, and make the alternate angles equal, AGH=GHD, and BGH=GHC; or make the internal angles BGH, GHD, on the same side of it, together equal to two right angles; AB is parallel to CD.



For since the angle BGE is equal to AGH (Prop. 15), it is also equal to the angle GHD; therefore AB is parallel to CD (2 Cor. Def.).

Again, because the angles BGH, GHD are together equal to two right angles, and the angles BGH, BGE are together equal to two right angles (Prop. 13); the angle BGE is equal to GHD; therefore AB is parallel to CD. Therefore, if a straight line &c. Q. E. D.

Otherwise. For if they were not parallel the alternate angles would not be equal, nor the two internal angles on the same side together equal to two right angles.

Cor. Two straight lines AB, CD, which are perpendicular to the same straight line EF, are parallel to each other.—*See the second figure.*

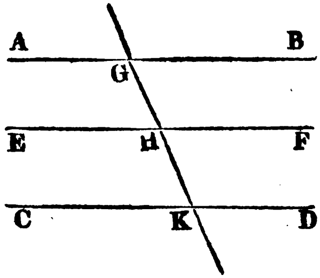
For the angles AEF, EFD are equal, because they are right angles. But they are alternate angles, therefore AB is parallel to CD.

PROPOSITION III. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

Let two straight lines AB, CD be parallel to the same straight line EF; they are parallel to each other.

Draw GK intersecting the three straight lines AB, EF, CD; then the angle AGH is equal to the alternate angle GHF (Prop. 1), and the angle HKD is equal to GHF (2 Cor. Def.); therefore the angle AGH is equal to the alternate angle HKD, therefore AB is parallel to CD (Prop. 2). Therefore straight lines &c. Q. E. D.



Bezout, an eminent mathematician, calls 2 Cor. Def. a proposition, and demonstrates it as follows.

PROPOSITION IV. THEOREM.

If a straight line intersect two parallel straight lines, the angles which it makes with them on the same side, are equal to each other.—*See figure, Prop.* 2.

Let EF intersect the parallel lines AB, CD; the angles BGE, DHE, or the angles AGH, CHF, which it makes with them, on the same side, are equal.

For the parallels AB, CD, having no inclination to each other, must be equally inclined, on the same side, to every line with which they are compared.

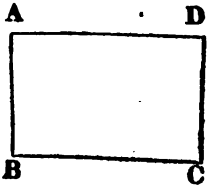
END OF BOOK I.

ELEMENTS OF GEOMETRY.

BOOK II.

Definitions and Preliminary Observations.

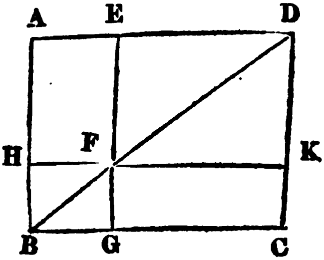
I. Every right angled parallelogram, or *rectangle*, is said to be contained by any two of the straight lines which are about one of the right angles. Thus, the right angled parallelogram AC is called the rectangle contained by AD and DC, or by AD and AB, &c.



For the sake of brevity, instead of the *rectangle contained* by AD and DC, we shall simply say the rectangle AD, DC, placing a point between the two sides of the rectangle. Instead of the square of a line, as AD, we may write AD^2.

$(AD + DC) . (AD - DC)$, or $\overline{AD + DC} . \overline{AD - DC}$, signifies the rectangle contained by the sum and difference of AD and DC.

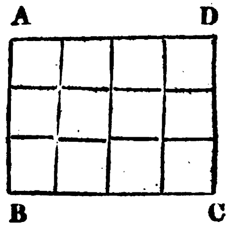
II. In every parallelogram either of the two parallelograms about a diameter, together with the two complements, are called a gnomon. Thus, the parallelogram HG, together with the complements AF, FC, are the gnomon of the parallelogram AC. This gnomon may also, for the sake of brevity, be called the gnomon AGK, or EHC.



All lines are measured by other lines. A certain line, arbitrarily assumed, is called *unity;* and the length of every other line is represented by the number of lines called *units,* which are contained in that line. Thus, if a line of one inch in length be called unity, the length of any other line as a cubit, is mea-

sured by the number of units or inches which it contains. If the cubit contains 21 lines, each an inch long, then the cubit is 21 inches long; or, more properly, the ratio of the line called a cubit to the line called an inch, is that of the number 21 to the number 1. In this manner lines are compared with lines. But lines cannot be compared with surfaces, nor with solids; because a line, a surface, and a solid are magnitudes of different kinds, and therefore cannot be compared together.—LUDLAM.

Surfaces are compared with, and are measured by other surfaces. If the length and breadth of a rectangle ABCD be each divided into inches, and if lines be drawn through all the points of division parallel to the sides of the rectangle, the surface of the rectangle will be divided into squares, whose sides will be one inch in length. The surface of the whole figure will be measured by the number of the squares (called square inches, or superficial inches) which it contains. Now the number of square inches in the surface will be found by multiplying the number of linear inches contained in one side of the rectangle by the number of linear inches contained in the other side. For there will be as many square inches in each row of squares as there are linear inches in one side of the rectangle, and as many such rows of squares as there are linear inches in the other side. This is manifest by inspection of the figure. Suppose the length of such a rectangle to be 4 inches and the breadth 3; then the whole surface contains 12 square inches; that is, the whole surface of this rectangle is to the surface of a square whose side is one inch, in the ratio of the number 12 to the number 1. In this manner are surfaces compared with surfaces; but they cannot be compared with lines nor solids. LUDLAM.



Hence we can find the area of any parallelogram, or of any plane triangle. All parallelograms on the same base and between the same parallels are equal (35. 1.), whether they be right or oblique angled. Let a and b denote the number of feet (or any other linear measure) in the base and altitude of any parallelogram, then will $a \times b$ denote the number of square feet in its surface. Again, a triangle is half of a parallelogram on the same base and between the same parallels (41. 1.), and all triangles on the same base and between the same parallels are equal (37. 1.); consequently $\frac{1}{2}ab$ will denote the number of square feet in any plane triangle. Hence the surface of any parallelogram is represented by the product

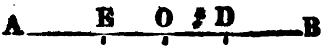
of the linear units in the base and altitude; and the surface of any plane triangle by half the same product.

Cor. If $a = b$, then AB = BC, and the rectangle becomes a square. In this case $a \times b = a \times a$, or a^2. Therefore the surface of a square is expressed by the second power of the number denoting its side. ED.

Hence it follows that such rectangles will be represented by the product of the multiplication of the number of lines, called units, in one side, by the number of lines, called units, in the other side. That is, whatever ratio any two rectangles have to each other, the same will be the ratio of the two products of the multiplication of the number of lines, called units, in their sides respectively. Hence rectangles in geometry and products in arithmetic are often put for each other, and the terms are applied promiscuously, though improperly. Consequently, what is proved concerning the equality of certain rectangles in the second book, will also be true of the products of the multiplication of the number of lines, called units, in their bases and altitudes. Hence, if the number of lines, called units, in the side of a rectangle, or the different parts into which the side is divided, be represented algebraically by letters, most of the propositions in the second book may be concisely and easily demonstrated by the rules of algebra. In the following propositions the letters $a, b, x,$ &c. may represent any numbers whatever, as well as lines. LUDLAM.

PROPOSITION A. THEOREM. ED.

Of any two unequal magnitudes, or quantities, if the sum and difference be added together, half the aggregate will be the greater quantity; and if the difference be subtracted from the sum, half the remainder will be the less quantity.



Let AD, DB be two unequal magnitudes; the greater magnitude AD is equal to half the aggregate of AD + DB and AD — DB; and the less magnitude DB is equal to half the remainder of (AD + DB) — (AD — DB).

Let AD, DB be placed in the same straight line AB. Bisect AB in O, then AO is the semi-sum of AD and DB. Make AE = BD, then DE is the difference of AD, DB, and OE = OD is the semi-difference of AD, DB. Now AO + OD = AD, the greater magnitude, and AO — OD = BO — OD = DB, the less. But $AO + OD = \frac{1}{2} AB + \frac{1}{2} DE = \frac{1}{2} (AB + DE)$

(Propor. 23), and AO — OD — ½ AB = ½ DE = ½ (AB — DE). Therefore, of any two &c. Q. E. D.

Cor. 1. If the semi-sum of two magnitudes be subtracted from the greater magnitude, the remainder is the semi-difference. For AD — AO = OD.

Cor. 2. The distance of the point of section from the middle of the line is equal to half the difference of the two segments. For DO = ½DE = ½(AD —DB).

Cor. 3. The greater segment is equal to the sum of half the line and the distance of the point of section from the middle of the line, and the less segment is equal to the difference between half the line and the distance of the point of section from the middle of the line. For AD = AO + OD, and BD = BO — OD.

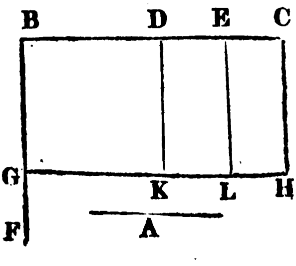
Algebraically. Let s denote the sum of any two unequal magnitudes or quantities, x the greater quantity, y the less, and d their difference; then will $x + y = s$, and $x - y = d$. Add the two equations together, then $2x = s + d$, $\therefore x = \frac{1}{2}(s + d)$, which is the first part of the prop.

Again, subtract the second equation from the first, then $2y = s - d$, $\therefore y = \frac{1}{2}(s - d)$, which is the second part.

PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the whole line and the several parts of the divided line.

Let A and BC be two straight lines, and let BC be divided into any parts in the points D, E; the rectangle A.BC is equal to the sum of the several rectangles A.BD, A.DE, A.EC, or A.BC = A.BD + A.DE + A.EC.



From B draw BF perp. to BC, and make BG equal to A; through G draw GH parallel to BC (31. 1), and through D, E, C draw DK, EL, CH parallel to BG; then BH, BK, DL, and EH are rectangles, and BH = BK + DL + EH.

But BH = BG.BC = A.BC, because BG = A; and BK = BG.BD = A.BD; and DL =

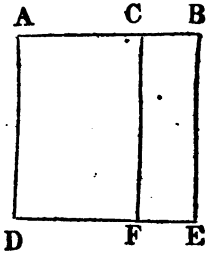
DK.DE = A.DE, because DK = BG = A (34. 1). In like manner EH = A.EC. Therefore A.BC = A.BD + A.DE + A.EC. Therefore, if there be two &c. Q. E. D.

Algebraically. Let s denote the whole line BC, and a, b, c, the several parts BD, DE, EC; then $s = a + b + c$. Let x denote the line A. Multiply the equation by x, then $sx = ax + bx + cx$, which is the prop.

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the sum of the rectangles contained by the whole line and each of the parts is equal to the square of the whole line.

Let the line AB be divided into any two parts in the point C; the sum of the rectangles AB.BC and AB.AC is equal to the square of AB, or AB.AC + AB.BC = AB².



On AB describe the square AE (46. 1), and through C draw CF parallel to BE (31. 1); then AF + CE = AE. But AF = AD.AC = AB.AC, because AD = AB; and CE = BE.BC = AB.BC; and AE = AB². Therefore AB.AC + AB.BC = AB². Therefore, if a straight line &c. Q. E. D.

Algebraically. Let s denote the whole line, and a and b the two parts; then $a + b = s$. Multiply this equation by s, then $as + bs = ss$, which is the prop.

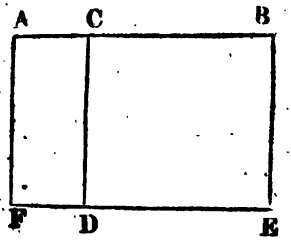
Scholium. This prop. is a particular case of the first; for if s be divided into the parts a and b, and x be equal to s, then $sx = ax + bx$ becomes $ss = as + bs$.

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square of that part together with the rectangle contained by the two parts.

Let the line AB be divided into two parts in the point C; the rectangle AB.BC is equal to the square of BC together with the rectangle AC.BC, or AB.BC = BC² + AC.CB.

On BC describe the square CE (46. 1); produce ED to F, and through A draw AF parallel to CD (31. 1); then AE = AD + CE. But AE = AB.BE = AB.BC, because BE = BC; and AD = AC.CD = AC.CB; and CE = BC^2. Therefore AB.BC = BC^2 + AC.CB. Therefore, if a straight line &c. Q. E. D.



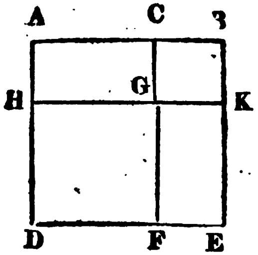
Algebraically. Let s denote the whole line, and a and b the two parts; then $s = a + b$. Multiply this equation by b, then $sb = ab + bb$, which is the prop.

PROPOSITION IV. THEOREM.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts together with twice the rectangle contained by the parts.

Let the line AB be divided into two unequal parts in the point C, or rather, let AC, BC be two unequal lines, and let AB be their sum; the square of AB is equal to the sum of the squares of AC, CB, with twice the rectangle contained by AC, CB; that is, $AB^2 = AC^2 + CB^2 + 2AC.CB$.

On AB and BC describe the squares AE and CK (46. 1). Produce CG and KG to meet the sides of the square AE in F and H. Now the square described on AB consists of four parts. The first part is the square CK described on BC; the second is FH or the square on AC, for CF and KH, being parallel to the sides of the square AE, are equal, therefore if the equal parts CG and KG be taken from them, the remainders GF and GH will be equal; and they are also equal to HD and DF (34. 1); wherefore the figure FH is equilateral. But the angle HDF is a right angle, therefore FH is a square (33 Def.).



The two figures AG, GE are each equal to a rectangle under AC, CB, because HG = GF, and KG = BC = CG. Hence HF + CK + AG + GE = AE, or $AB^2 = AC^2 + CB^2 + 2AC.CB$. Wherefore, if a straight line &c. Q. E. D. Ed.

Cor. 1. From the demonstration it is manifest that the parallelograms about the diagonal of a square are squares. Suppose BD to be drawn, then it has been proved that CK and HF are squares.

Cor. 2. The square of the sum of two lines, AC, CB, is greater than the sum of their squares by twice the rectangle contained by those lines. For this is the prop. in different words. Ed.

Cor. 3. If a straight line be divided into two equal parts, the square of the whole line will be equal to four times the square of half the line.

If $AC = CB$, then
$AB^2 = AC^2 + AC^2 + 2AC.AC = 4AC^2$. Ed.

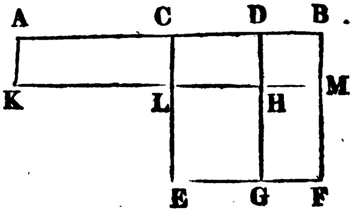
Algebraically. Let s denote the whole line, and a and b the two parts; then $s = a + b$. Square both sides of the equation, then $ss = aa + 2ab + bb$, which is the prop.

Cor. If the two parts be equal, or $a = b$, then $ss = 4aa$, that is, the square of a line, or of any number, is four time the square of its half.

PROPOSITION V. THEOREM.

If a straight line be divided into two unequal parts, the rectangle contained by their sum and difference is equal to the difference of their squares.

Let the straight line AD be divided into two unequal parts in the point C, and let AC be the greater part; the rectangle $(AC + CD).(AC - CD)$ is equal to $AC^2 - CD^2$.



Produce AD, and take $CB = AC$; then $AD = AC + CD$, and $DB = AC - CD$. On CB describe the square CF; through D draw DG parallel to BF; take $BM = BD$; draw MK parallel to AB, and AK parallel to CE. Then LG is a square (33 Def.), and is equal to CD^2.

Because $CH = HF$ (43. 1), $CM = DF$; and because $AL = CM$ (36. 1), $AL = DF$, therefore $AH = CH + DF =$ gnomon CMG. But $AH = AD . DH = AD . DB$, because $DH = DB$ (1 Cor. 4. 2); therefore the gnomon $CMG = AD . DB$. But

$CMG = CF - LG = AC^2 - CD^2$; therefore $AD.DB = AC^2 - CD^2$. Wherefore, if a straight line &c. Q. E. D.

Cor. 1. The rectangle contained by the sum and difference of two unequal straight lines AC, CD, is equal to the difference of their squares; that is, $(AC + CD).(AC - CD) = AC^2 - CD^2$. This cor. is the prop. in different words.

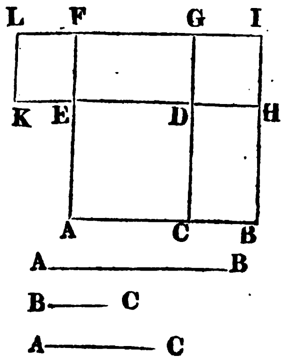
Cor. 2 If a straight line, or number, be divided into any two unequal parts, and into other two unequal parts, &c. in pairs; then, of all the rectangles contained by any pairs of segments the greatest is the square described on half the line.

For $AD.DB = AC^2 - CD^2$, therefore the rect. $AD.DB$ is always less than AC^2 by CD^2, and constantly increases as DC^2 decreases, till the point D coincide with C, and then $AD.DB$ becomes $AC.CB$, or AC^2. Ed.

PROPOSITION VII. THEOREM.

The square of the difference of any two unequal straight lines is less than the sum of their squares by twice the rectangle contained by those lines.

Let AB and BC be two unequal lines, of which AB is the greater, and let AC be their difference; the square of AC is less than the sum of the squares of AB and BC by twice the rectangle under AB and BC, that is, $AC^2 = AB^2 + BC^2 - 2AB.BC$.



Place AC on AB; on AB and AC describe the squares AI and AD (46. 1); produce AE and CD to F and G; produce ED and FG to H and I, and indefinitely to K and L; in IL take $FL = BC$, and complete the square LE. The sides of the rectangle BG are equal to AB, BC; and the sides of the rectangle GK are also equal to AB, BC, for $GF = AC$ (34. 1), and FL or $LK = BC$; therefore $GL = AB$, and LK or $GD = BC$. Therefore the rectangles BG and GK are equal to each other, and to the rectangle under AB, BC. Therefore $BG + GK = 2AB.BC$. Now if $BG + GK$ be taken from the

whole figure ABILKE, there will remain AD, that is, $AC^2 = AB^2 + BC^2 - 2AB . BC$. Therefore, the square &c. Q. E. D.

Cor. 1. If a straight line AB be divided into any two parts AC, CB, the squares of the whole line and of one of the parts are together equal to twice the rectangle contained by the whole line and that part together with the square of the other part. That is, $AB^2 + BC^2 = 2AB . BC + AC^2$, by adding $2AB . BC$ to the two equals in the prop.

Cor. 2. Hence, the sum of the squares of any two unequal lines is equal to twice the rectangle contained by those lines together with the square of their difference.

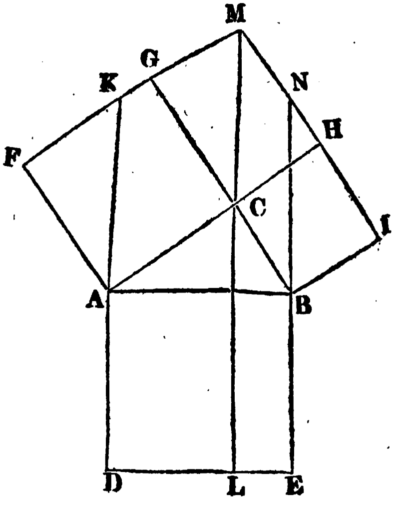
Algebraically. Let a denote the greater of any two unequal lines, b the less, and d the difference; then $d = a - b$, and $dd = (a-b)^2 = aa - 2ab + bb$.

PROPOSITION B. THEOREM.

This is Prop. 47. 1.

In any right angled triangle the square described on the hypothenuse is equal to both the squares described on the other two sides.

Let ABC be a right angled triangle, having the right angle ACB, and let the squares AE, FC, CI be described on the three sides AB, AC, BC; AE = FC + CI.



Through C draw ML parallel to AD, meeting FG produced in M, and produce DA to meet FG in K. Because BAD is a right angle, its adjacent angle BAK is a right angle (13. 1). From the right angles BAK, CAF take the common angle CAK, and there remain the equal angles BAC, FAK. Consequently the triangles BAC, FAK, having the side AC = AF, are equal in all respects (A. 1). Thus, AB = AK, or AD = AK; therefore the parallelograms AL, AM, which are on equal bases AD, AK, and between the

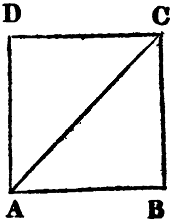
same parallels DK, LM, are equal (36. 1). But the square FC is equal to the parallelogram AM, because they stand on the same base AC and between the same parallels AC, FM (35. 1). Consequently the paral. AL = square FC. In the same manner it may be demonstrated that the paral. BL is equal to the square CI. Hence AE = CF + CI. Therefore, in any &c. Q. E. D. WEST.

COR. 1. The square described on one of the sides of a right angled triangle is equal to the difference of the squares described on the hypothenuse and the other side.

For $AC^2 + BC^2 = AB^2$. From these equals take BC^2, then $AC^2 = AB^2 - BC^2$. ED.

COR. 2. If a square AE be described on the hypothenuse AB of a right angled triangle ABC, and a perpendicular CL from the right angle C meet the opposite side DE of the square AE, the perpendicular will divide the square into two rectangles AL, BL, which are equal to the squares of the sides AC, BC adjacent to them. ED.

COR. 3. If two right angled triangles have two sides of one triangle equal to two corresponding sides of the other, they are equal in all respects.



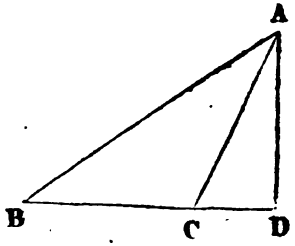
COR. 4. The square described on the diagonal of a square is double of that square.

Let ABCD be a square, and AC its diagonal. Because the triangle ABC is right angled and isosceles, $AC^2 = AB^2 + BC^2 = 2AB^2$.

Scholium. Because $AC^2 = 2AB^2$, $AC^2 : AB^2 :: 2 : 1$, $\therefore AC : AB :: \sqrt{2} : 1$. Now since the value of $\sqrt{2}$ cannot be exactly expressed in numbers, $\sqrt{2}$ and 1 are said to be incommensurable quantities; consequently the diagonal of a square is incommensurable with its side; that is, the side and the diagonal of a square have no finite ratio.

PROPOSITION XII. THEOREM.

In any obtuse angled triangle if a perpendicular be drawn from either of the two acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the sum of the squares of the two sides containing it by twice the rectangle contained by the base and its external part intercepted between the perpendicular and the obtuse angle.



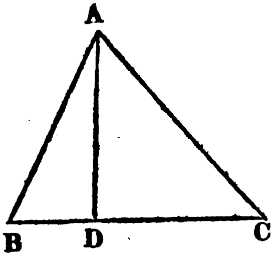
Let ABC be a triangle, having an obtuse angle ACB, and from the point A let AD be drawn perp. to BC produced (12. 1); the square of AB is greater than the squares of AC, CB, by twice the rectangle BC.CD; or $AB^2 = BC^2 + AC^2 + 2BC.CD$.

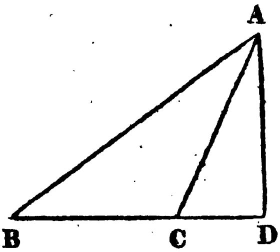
For $BA^2 = BD^2 + DA^2$ (B.2) or $\overline{BC + CD}^2 + AD^2$, because $BD = BC + CD$. But $\overline{BC + CD}^2 = BC^2 + 2BC.CD + CD^2$ (4. 2). Hence $BA^2 = BC^2 + 2BC.CD + CD^2 + DA^2$. But $CD^2 + DA^2 = CA^2$; therefore, by putting equals for equals, $BA^2 = BC^2 + 2BC.CD + CA^2$ or $BC^2 + AC^2 + 2(BC.CD)$. Therefore, in any &c. Q. E. D.

PROPOSITION XIII. THEOREM.

In any triangle the square of the side subtending an acute angle is less than the sum of the squares of the two sides containing that angle, by twice the rectangle contained by either of those sides and the distance between the acute angle and the perpendicular drawn from the opposite angle to that side.

Let ABC be any triangle, and the angle at B one of its acute angles; on BC, one of the sides containing B, let fall the perp. AD from the opposite angle A; the square of AC, opposite to B, is less than the squares of CB, BA, by twice the rectangle CB.BD, or $AC^2 = AB^2 + BC^2 - 2BC.BD$.





Because the angle at D is right, $AC^2 = AD^2 + DC^2$ (B. 2). But $AD^2 = AB^2 - BD^2$ (1 Cor. B. 2), and $DC^2 = \overline{BC - BD}^2$, or $\overline{BD - BC}^2$, according as the angle ACB is acute or obtuse, that is, $DC^2 = BC^2 + BD^2 - 2BC.BD$ (7. 2). Therefore, by adding the last two equals, $AD^2 + DC^2$, or $AC^2 = AB^2 + BC^2 - 2BC.BD$. Therefore, in any triangle &c. Q. E. D. Ed.

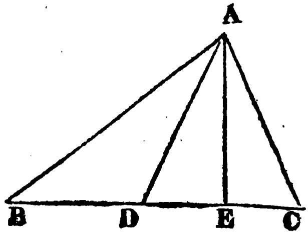
Scholium. From Prop. B, 12 and 13 it is manifest, that any angle of a triangle is right, obtuse, or acute, according as the square of the side opposite to it is equal to, greater, or less than the sum of the squares of the sides containing it. Ed.

PROPOSITION C. THEOREM.

If a straight line be drawn from any angle of a triangle to the middle of the opposite side, the sum of the squares of the other two sides will be double of the sum of the squares of that line and of half the base.

Let ABC be a triangle, and let any side BC be bisected in D, and from D let DA be drawn to the opposite angle; the squares of BA and AC are together double of the squares of BD and DA, or $BA^2 + AC^2 = 2BD^2 + 2\,DA^2$.

From A draw AE perp. to BC, then $AB^2 = BD^2 + DA^2 + 2BD.DE$ (12. 2), or $DC^2 + DA^2 + 2DC.DE$, and $AC^2 = DC^2 + DA^2 - 2DC.DE$ (13. 2); therefore, by adding these equals, $AB^2 + AC^2 = 2DC^2 + 2DA^2$. Therefore, if a straight line &c. Q. E. D.

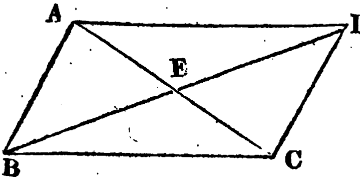


PROPOSITION D. THEOREM.

The sum of the squares of the two diagonals of any parallelogram is equal to the sum of the squares of the four sides.

Let ABCD be a paral., of which the diagonals are AC and BD; the sum of the squares of AC and BD is equal to the sum of the squares of AB, BC, CD, DA.

Let AC and BD intersect each other in E, then they will bisect each other in E (I. 1). Since BD is bisected in E, $AB^2 + AD^2 = 2BE^2 + 2AE^2$ (C. 2); and for the same reason $CD^2 + CB^2 = 2BE^2 + 2EC^2 = 2BE^2 + 2AE^2$, because $EC = AE$; therefore $AB^2 + AD^2 + DC^2 + BC^2 = 4BE^2 + 4AE^2$. But $4BE^2 = BD^2$ (3 Cor. 4. 2), and $4AE^2 = AC^2$, because BD and AC are both bisected in E; therefore $AB^2 + AD^2 + CD^2 + BC^2 = BD^2 + AC^2$. Therefore, the sum &c. Q. E. D.

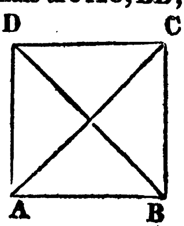


PROPOSITION E. THEOREM. ED.

The sum of the squares of the two diagonals of a square is equal to the sum of the squares of the four sides.

Let ABCD be a square, of which the diagonals are AC, BD; the sum of the squares of AC, BD is equal to the sum of the squares of AB, BC, CD, DA, or to $4AB^2$.

For $BD^2 = AB^2 + AD^2 = 2AB^2$ (B. 2), and $AC^2 = AB^2 + BC^2 = DC^2 + BC^2 = 2AB^2$. Consequently, by adding equals to equals, $AC^2 + BD^2 = DC^2 + BC^2 + AB^2 + AD^2 = 4AB^2$. Therefore, the sum &c. Q. E. D.

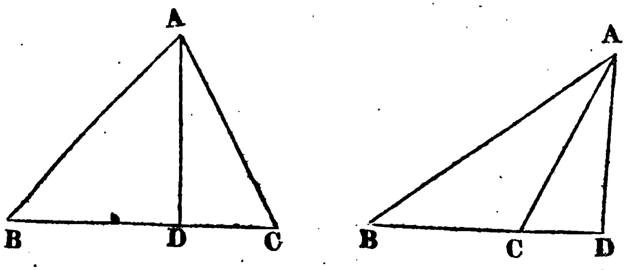


PROPOSITION F. THEOREM. ED.

If a perpendicular be drawn from any angle of a triangle to the opposite side, the rectangle contained by

the sum and difference of the other two sides is equal to the rectangle contained by the sum and difference of the segments of the base made by the perpendicular.

Let ABC be a triangle, having the side AB greater than AC, and AD a perp. from the angle A to the opposite side BC; then $\overline{AB + AC}.\overline{AB - AC} = \overline{BD + DC}.\overline{BD - DC}$.



For $AB^2 = BD^2 + DA^2$ (B. 2),
and $AC^2 = DC^2 + DA^2$;
therefore the differences of those two equals are equal, or $AB^2 - AC^2 = BD^2 - DC^2$.
But $AB^2 - AC^2 = \overline{AB + AC} . \overline{AB - AC}$ (5. 2),
and $AD^2 - DC^2 = \overline{BD + DC} . \overline{BD - DC}$;
therefore $\overline{AB + AC}.\overline{AB - AC} = \overline{BD + DC}.\overline{BD - DC}$.
Therefore, if a perp. &c. Q. E. D.

Cor. 1. If a perpendicular be drawn from any angle of a triangle to the opposite side, the difference of the squares of the other two sides is equal to the difference of the squares of the segments of the base made by the perpendicular. For, by the dem. $AB^2 - AC^2 = BD^2 - DC^2$.

Cor. 2. If a perpendicular be drawn from any angle of a triangle to the opposite side or base, the difference between the squares of the other two sides, or the rectangle under their sum and difference, is equal to twice the rectangle contained by the base and the distance of the perpendicular from the middle of the base.—*(See Fig. of Prop. C.)*.

Let D be the middle of the base BC, and let AF be perp. to BC; then $AB^2 - AC^2 = BF^2 - FC^2$ (Cor. 1), or $(BF + FC).(BF - FC)$. But $BF - FC = DF + DC - FC = 2DF + FC - FC = 2DF$. Consequently $AB^2 - AC^2 = BC . 2DF = 2(BC . DF)$, or $(AB + AC).(AB - AC) = 2(BC . DF)$. Ed.

Cor. 3. If a perpendicular be drawn from any angle of a triangle to the opposite side, that side will be to the sum of the other two sides, as half their difference is to the distance of the perpendicular from the middle of the base.

For $2BC . DE = (AB + AC) . (AB - AC)$, therefore $2BC : AB + AC :: AB - AC : DE$ (Propor. 33), therefore $BC : AB + AC :: \frac{1}{2}(AB - AC) : DE$ (Propor. 30). Ed.

Scholium. If the perpendicular be always drawn from the greatest angle of any triangle to the opposite side, the proposition will be reduced to one case only, and its application to practice will be more convenient. The proposition will then be expressed as follows.

If a perpendicular be drawn from the greatest angle of a triangle to the opposite side, the greatest side will be to the sum of the other two sides, as their difference is to the difference of the segments of the greatest side made by the perpendicular.

By means of this proposition, or Cor. 2 or 3, we can divide a triangle, of which all the sides are given, into two right angled triangles, by determining the segments of the base, and thence the position and length of the perpendicular. Thus, by the proposition, half the sum of the base and the fourth proportional is equal to the greater segment (A. 2), and half the difference between the base and the fourth proportional is equal to the less segment. Ed.

THE PRINCIPAL THEOREMS IN BOOK II.

Of any two unequal magnitudes, or quantities, if the sum and difference be added together, half the aggregate will be the greater quantity; and if the difference be subtracted from the sum, half the remainder will be the less quantity.

Note.—This prop. may be briefly expressed by signs as follows.

Of two unequal quantities half the sum + half the difference is equal to the greater quantity, and half the sum — half the difference is equal to the less.

If there be two straight lines, and if one of them be divided into any number of parts, the sum of the rectangles contained by the whole line and the several parts of the divided line is equal to the rectangle contained by the two whole lines.

If a straight line be divided into any two parts, the two rectangles contained by the whole line and each of the parts are together equal to the square of the whole line.

If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the rectangle contained by the two parts together with the square of the foresaid part.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts together with twice the rectangle contained by the two parts. Or, in other words, the square of the sum of two lines is greater than the sum of their squares by twice the rectangle contained by those lines.

If a straight line be divided into two equal parts, the square of the whole line is equal to four times the square of half the line.

The rectangle contained by the sum and difference of two lines is equal to the difference of their squares.

The square of the difference of any two lines is less than the sum of their squares by twice the rectangle contained by those lines.

In any right angled triangle the square described on the hypothenuse is equal to both the squares described on the other two sides.

If two right angled triangles have two sides of one triangle equal to two corresponding sides of the other, they are equal in all respects.

In any obtuse-angled triangle, if a perpendicular be drawn from either of the two acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the sum of the squares of the sides containing it by twice the rectangle under the base and the distance between the perpendicular and the obtuse angle.

In any triangle the square of the side subtending an acute angle is less than the sum of the squares of the two sides containing that angle by twice the rectangle under either of these sides and the distance between the acute angle and the perpendicular drawn from the opposite angle to that side.

If a straight line be drawn from any angle of a triangle to the middle of the opposite side, the sum of the squares of the other two sides will be double of the sum of the squares of that line and of half the base.

The sum of the squares of the two diagonals of any parallelogram is equal to the sum of the squares of the four sides.

If a perpendicular be drawn from the greatest angle of a triangle to the opposite side, the rectangle contained by the sum and difference of the other two sides is equal to the rectangle contained by the sum and difference of the segments of the base made by the perpendicular; or, the greatest side is to the sum of the other two sides, as their difference is to the difference of the segments of the greatest side made by the perpendicular.

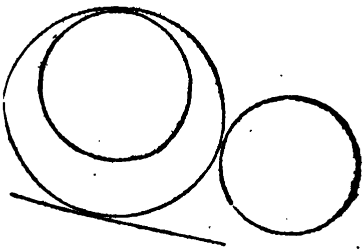
If a perpendicular be drawn from any angle of a triangle to the opposite side, the difference of the squares of the other two sides is equal to the difference of the squares of the segments of the base made by the perpendicular.

END OF BOOK II.

ELEMENTS OF GEOMETRY.

BOOK III.

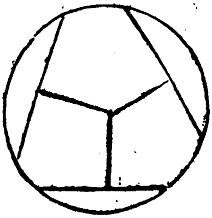
DEFINITIONS.*



1. A straight line is said to touch a circle, when it meets the circle, and, being produced, does not cut it. Such a line is usually called a tangent.

A. A straight line drawn from any point without a circle to the concave part of the circumference is called a secant. Ed.

2. Circles are said to touch one another when they meet, but do not cut one another.



3. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

4. And the straight line on which the greater perpendicular falls is said to be farther from the centre.

B. An arch of a circle is any part of the circumference.

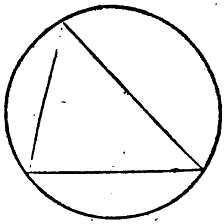
And the chord of any arch is the straight line joining the two extremities of the arch. Ed.

5. A segment of a circle is the figure contained by a straight line and the arch which it cuts off, or, it is the space included between an arch and its chord.

---

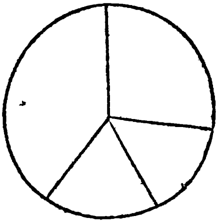
* Numbers 1, 2, 3, 4, 7, are explanations, not definitions. Ed.

6. An angle in a segment of a circle is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the chord which is the base of the segment.



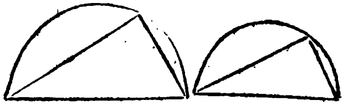
7. An angle is said to stand on the arch intercepted between the straight lines which contain the angle.

8. The sector of a circle is the figure contained by two straight lines drawn from the centre and the arch of the circumference between them.



C. If the radii, or the two straight lines containing the angle, be perpendicular to each other, or form a right angle, the sector is called a quadrant. Ed.

9. Similar segments of circles are those which contain equal angles.



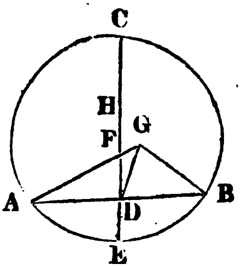
PROPOSITION I. PROBLEM.

To find the centre of a given circle.—*See Note.*

Let ABC be the given circle; it is required to find its centre.

Draw any chord AB, and bisect it in D; from the point D draw DC perp. to AB, and produce it to E, and bisect CE in F; the point F is the centre of the circle ABC.

For, if F be not the centre of the circle, some other point either in the line CE, or out of it, must be the centre. Suppose the centre to be in the line CE, at the point H different from F. Then the radii HE, HC are equal. But the straight lines FC, FE are equal (by construction); and HC is less than FC (9 Ax.); therefore HC is less than FE. Much more then is HC less than HE. But HC and HE are supposed to be equal, which is impossible. Consequently the point H cannot be the centre of the circle.



Nor can the centre be a point out of the line CE. For, if possible, let G be the centre, and join GA, GD, GB; then, because DA is equal to DB, and DG is common to the two triangles ADG, BDG, and the base GA is equal to GB, because they are radii of the same circle, the angle ADG is equal to GDB (8. 1); therefore the angle GDB is a right angle. But FDB is also a right angle; wherefore the angle FDB is equal to GDB, the greater to the less, which is impossible. Therefore G is not the centre of the circle ABC.

Now since it has been proved that the centre of the circle is not in the line CE, at any point H different from F, nor out of CE, at any point G, the centre must be at the point F; that is, F is the centre of the circle. Which was to be found.*

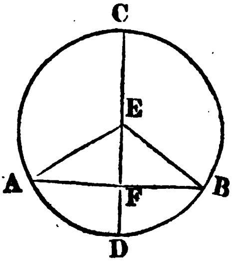
Cor. From this demonstration it is manifest, that if a straight line in a circle bisect another at right angles, the centre of the circle is in the line which bisects the other; and that if the bisecting line be bisected, the point of section will be the centre of the circle.

PROPOSITION III. THEOREM.

In a circle if a radius or a diameter bisect a chord, it will cut the chord at right angles; and if it cut a chord at right angles, it will bisect the chord.

1. Let ABC be a circle, and let a diameter CD bisect any chord AB in the point F; CD cuts AB at right angles.

Take E the centre of the circle (1. 3), and join EA, EB. Then, because AF is equal to FB, and FE common to the two triangles AFE, BFE, and the base EA is equal to EB, the angle AFE is equal to BFE (8. 1), therefore each of the angles AFE, BFE is a right angle; wherefore the diameter CD, bisecting the chord AB, cuts it at right angles.



2. Again, let CD cut AB at right angles; CD also bisects AB.

* The demonstration of this prop. in Playfair's Geometry is defective.

The same construction being made, because the radii EA, EB are equal, the angle EAF is equal to EBF. But the right angles AFE, BFE are equal. Hence the angle AEF is equal to BEF (2 Cor. 32. 1). But the side EF is common to the two triangles EAF, EBF; therefore the other sides are equal (A. 1); therefore AF is equal to FB. Wherefore, in a circle &c. Q. E. D.

Cor. If a radius, or a diameter, bisect a chord, it will also bisect the arch subtended by that chord.

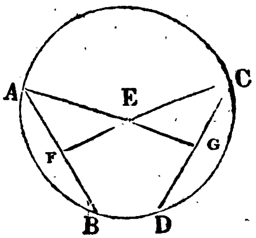
For the radii EA, EB are equal, and also the angles AED, BED; therefore if the sector AED be supposed to revolve round the radius ED as an axis of motion, till the point A fall on B, and the radius EA coincide with EB, it is evident that the arch AD will coincide with BD; therefore the arch AD is equal to BD, and the arch ADB is bisected by the radius ED. Ed.

PROPOSITION XIV. THEOREM.

Equal chords in a circle are equally distant from the centre; and chords in a circle which are equally distant from the centre are equal to one another.

1. Let the chords AB, CD, in the circle ABDC, be equal to each other; they are equally distant from the centre.

Find the centre E of the circle, and from E draw EF, EG perp. to the chords AB, CD; join AE, EC. The right angled triangles AEF, CEG, having the side AE equal to CE, and AF equal to CG (3. 3), are equal in all respects (3 Cor. B. 2). Thus, the perpendiculars EF, EG are equal, and therefore the chords AB, CD are equally distant from the centre E.

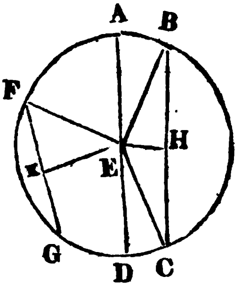


2. Next, let the chords AB, CD be equally distant from the centre of the circle, that is, let the perpendiculars EF, EG be equal; AB is equal to CD.

The right angled triangles AEF, CEG, having the sides AE, EF equal to the sides CE, EG, each to each, are equal in all respects. Thus, AF is equal to CG, and therefore the doubles of these, or the chords AB, CD, are equal (3. 3). Therefore, equal chords &c. Q. E. D. Ed.

PROPOSITION XV. THEOREM.

The diameter is the greatest straight line in a circle; and of all other lines that which is nearer to the centre is greater than one more remote; and the greater line is nearer to the centre than the less.



1. Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD is greater than any line BC, which is not a diameter, and BC is greater than FG.

From the centre draw EH perp. to BC, and EK perp. to FG; and join EB, EC, EF. Because AE is equal to EB, and ED to EC, AD is equal to EB and EC together. But EB, EC together are greater than BC (20. 1); wherefore AD is greater than BC.

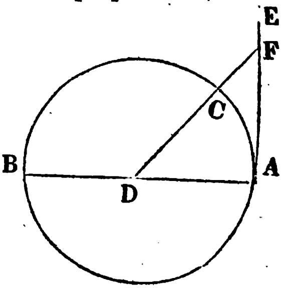
Again, because BC is nearer to the centre than FG, EH is less than EK (4 Def. 3). Now BC is double of BH (3. 3), and FG is double of FK; and the squares of EH, HB together are equal to the squares of EK, KF together, because they are equal to the squares of the radii EB, EF (B. 2). But the square of EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK; therefore the line BH is greater than FK; therefore BC is greater than FG.

2. Next, let BC be greater than FG; BC is nearer to the centre than FG; that is, EH is less than EK. Because BC is greater than FG, BH is greater than FK. But the squares of BH, HE together are equal to the squares of FK, KE together, and the square of BH is greater than the square of FK; therefore the square of EH is less than the square of EK; therefore the line EH is less than EK, that is, BC is nearer to the centre than FG. Wherefore, the diameter &c. Q. E. D.

PROPOSITION XVI. THEOREM.

A straight line drawn perpendicular to the diameter of a circle, from one extremity of it, is a tangent to the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB; and let AE be drawn from A perp. to AB; AE will touch the circle.



In AE take any point F, and join DF, cutting the circle in C. Because DAF is a right angle, it is greater than the angle AFD (3 Cor. 32. 1); therefore DF is greater than DA or DC (19. 1); therefore the point F is without the circle, and F is any point whatever in the line AE; therefore AE falls without the circle, and meets it only in the point A. Therefore AE touches the circle. Therefore, a straight line &c. Q. E. D.

Otherwise. Because DA is perp. to AE, it is the shortest distance of the point D from AE (C. 1); therefore any other point in AE is farther from D that A is, and consequently falls without the circle. Therefore AE falls without the circle, and meets it only in A.

Therefore, a straight line &c. Q. E. D. Ed.

PROPOSITION XVII. PROBLEM.

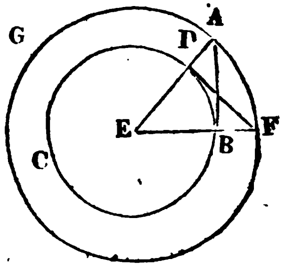
To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle. That is, to draw a tangent to a circle.

First, let A be a given point without the given circle BCD; it is required to draw a straight line from A which shall touch the circle.

Find the centre E of the circle (1. 3), and join AE; from the centre E, at the distance EA, describe the circle AFG; from the point D draw DF perp. to EA, and join EBF and AB; then AB touches the circle BCD.

Because E is the centre of the circles BCD and AFG, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two sides FE, ED, each to each; and they

contain the angle at E common to the two triangles AEB, FED; therefore the angle EBA is equal to EDF. But EDF is a right angle, wherefore EBA is a right angle. Now EB is drawn from the centre; therefore AB touches the circle (16. 3); and it is drawn from the given point A. Which was to be done.



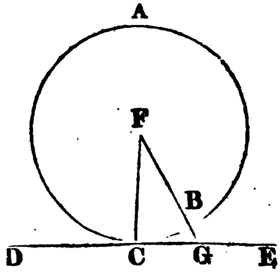
But if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF perp. to DE; then DF touches the circle (16. 3).

PROPOSITION XVIII. THEOREM.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact is perpendicular to the tangent.

Let the straight line DE touch the circle ABC in the point C; find the centre F, and join FC; FC is perp. to DE.

For, if FC be not perp. to DE, from the point F draw FBG perp. to DE; then because FGC is a right angle, GCF must be an acute angle (3 Cor. 32. 1); therefore FC is greater than FG (19.1). But FC is equal to FB; therefore FB is greater than FG; a part greater than the whole, which is impossible; wherefore FG is not perp. to DE. In the same manner it may be shown that no other line but FC can be perp. to DE; therefore FC is perp. to DE. Therefore, if a straight line &c. Q. E. D.

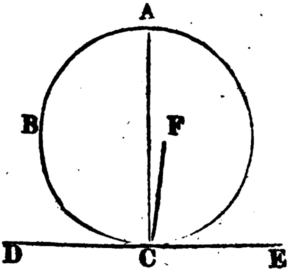


PROPOSITION XIX. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn perpendicular to the tangent, the centre of the circle is in that line.

Let the straight line DE touch the circle ABC in the point C, and from C let CA be drawn perp. to DE; the centre of the circle is in CA.

For, if not, let F be the centre, if possible, and join CF. Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perp. to DE (18. 3); therefore FCE is a right angle. But ACE is also a right angle; therefore the angle FCE is equal to ACE, the less to the greater, which is impossible. Wherefore F is not the centre of the circle ABC. In like manner it may be shown that no other point which is not in CA is the centre; that is, the centre is in CA. Therefore, if a straight line &c. Q. E. D.

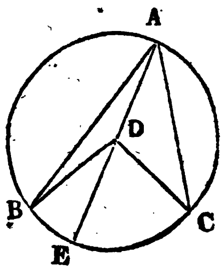


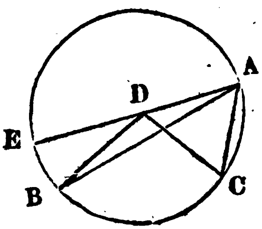
PROPOSITION XX. THEOREM.

The angle at the centre of a circle is double of the angle at the circumference, both standing on the same arch.

Let ABC be a circle, and BDC an angle at the centre, and BAC an angle at the circumference, both standing on the same arch BC; the angle BDC is double of BAC.

First, let D, the centre of the circle, be within the angle BAC; through D draw ADE. Because DA is equal to DB, the angle DAB is equal to DBA; therefore the angles DAB, DBA together are double of the angle DAB. But the angle BDE is equal to both the angles DAB, DBA (32. 1); therefore the angle BDE is double of DAB. For the same reason the angle CDE is double of CAD. Therefore the whole angle BDC is double of the whole angle BAC.

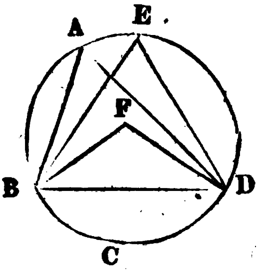




Again, let D, the centre of the circle, be without the angle BAC. Through D draw ADE. It may be demonstrated, as in the first case, that the angle CDE = 2CAE or 2CAB + 2BAE, and the angle BDE = 2BAE. Take the latter equals from the former, then the angle BDC = 2CAB. Therefore, the angle &c. Q. E. D.* Ed.

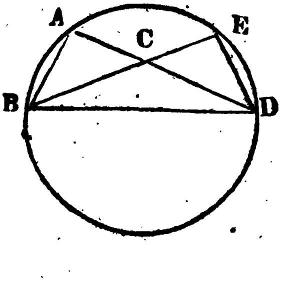
PROPOSITION XXI. THEOREM.

The angles at the circumference of a circle, standing on the same arch, are equal to one another.



Let ABCD be a circle, and BAD, BED, angles at the circumference, standing on the same arch BCD; the angles BAD, BED are equal to each other.

Take F, the centre of the circle. Let the segment BAED be greater than a semicircle, and join BF, FD. The angle BFD at the centre is double of each of the angles BAD, BED at the circumference, which stand on the same arch BCD (20. 3); therefore the angle BAD is equal to BED (7 Ax.).



Again, let the segment BAED be not greater than a semicircle, and let BAD, BED be angles in it; they are equal to each other.

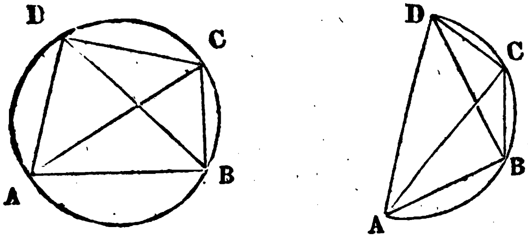
The angles BCA, DCE are equal (15. 1), and the angles ABC, EDC are equal by the former case; therefore the angles BAD, BED are equal (2 Cor. 32. 1). Therefore, the angles &c. Q. E. D. Ed.

* Euclid's demonstration of this case is faulty, for he assumes a prop. which has not been proved, and does not properly belong to Geometry.

PROPOSITION XXII. THEOREM.

The two opposite angles of any quadrilateral figure described in a circle, or in a segment of a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure described in the circle, or segment, ABCD; any two of its opposite angles ABC, ADC are together equal to two right angles.

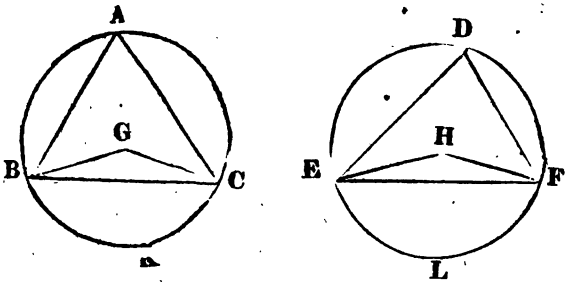


Draw the diagonals AC, BD. The angles CAB, CDB are equal, because they stand on the same arch BC (21. 3); and the angles ACB, ADB are equal, because they stand on the same arch AB; therefore the whole angle ADC is equal to the two angles CAB, ACB. To each of these equals add the angle ABC, then the two angles ABC, ADC are equal to the three angles ABC, CAB, BCA. But these are together equal to two right angles (32. 1); therefore the angles ABC, ADC are together equal to two right angles. In the same manner the angles BAD, BCD may be shown to be equal to two right angles. Therefore, the opposite angles &c. Q. E. D.

PROPOSITION XXVI. THEOREM.

In equal circles, if the angles at the centres or at the circumference be equal, the arches on which they stand are also equal.

Let ABC, DEF be equal circles, and let BGC, EHF be equal angles at their centres G, H; and BAC, EDF equal angles at their circumferences; the arch BKC is equal to the arch ELF.



Draw BC, EF. Let the circle ABC be applied to DEF, so that the centre G may lie on H, and the radius GB on HE; then the radius GC will coincide with its equal HF, because the angle BGC is equal to EHF. Now it is evident that the circle ABC will coincide with the equal circle DEF, and that the arch BKC will coincide with ELF, because the point B lies on E, and the point C on F.

Again, if the equal angles A and D at the circumferences be less than right angles, then, because the angles G and H at the centres are double of the equal angles A and D (20. 3), they are equal to each other, and therefore (by the first case) they stand on equal arches; therefore the angles A and D stand on equal arches. CRESSWELL.

But if the angles A and D be not less than right angles, let them be bisected; then, as before, the arches which subtend their halves are equal, each to each; therefore the whole arches which subtend the whole angles are equal. Wherefore, in equal circles &c. Q. E. D. CRESSWELL.

COR. 1. In the same manner it may be proved, that in the same circle equal angles stand on equal arches.

COR. 2. It is manifest that in the same circle, or in equal circles, the greater angle stands on the greater arch.

COR. 3. In the same circle, or in equal circles, equal chords are subtended by equal arches.

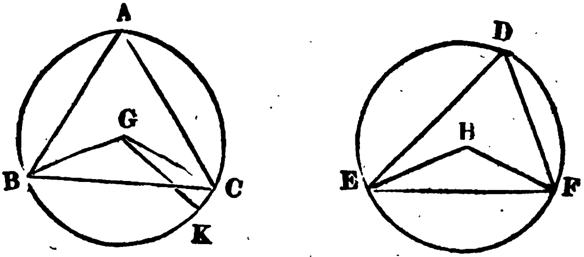
COR. 4. If a radius ED (Figure 3. 3) bisect a chord AB, it will also bisect the arch subtended by that chord. For the triangles AEF and BEF, being mutually equilateral, are equal in all respects; therefore the angle AEF = BEF, therefore, the arch AD = BD. ED.

PROPOSITION XXVII. THEOREM.

In equal circles the angles which stand on equal

arches are equal to one another, whether they be at the centres or at the circumferences of the circles.

Let the angles BGC, EHF at the centres, and the angles BAC, EDF at the circumferences of the equal circles ABC, DEF, stand on the equal arches BC, EF; the angle BGC is equal to EHF, and the angle BAC to EDF.



If the angle BGC be equal to EHF, the angle A is equal to D (20. 3). But, if not, one of them is the greater. Let BGC be the greater, and at the point G, in the line BG, make the angle BGK equal to EHF (23. 1); then the arch BK is equal to the arch EF (26. 3). But the arch EF is equal to the arch BC; therefore also the arch BK is equal to the arch BC, the less to the greater, which is impossible. Therefore the angle BGC is not unequal to EHF, that is, BGC is equal to EHF. But the angle A is half of the angle BGC (20. 3), and the angle D is half of the angle EHF; therefore the angle A is equal to the angle D. Wherefore, in equal circles &c. Q. E. D.

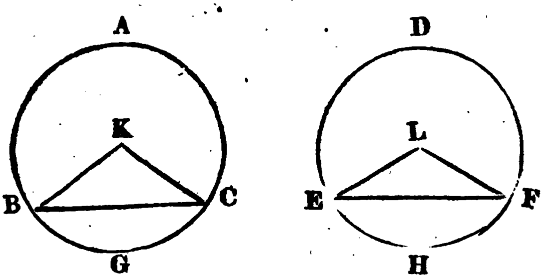
Cor. 1. In the same circle the angles which stand on the same arch, or on equal arches, are equal to one another. Ed.

Cor. 2. Hence it is evident that the angle which stands on the greater arch is the greater. Ed.

PROPOSITION XXVIII. THEOREM.

In equal circles equal chords cut off equal arches, the greater arch equal to the greater, and the less arch equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal chords in them, which cut off the two greater arches BAC, EDF, and the two less arches BGC, EHF; the greater arch BAC is equal to the greater EDF, and the less arch BGC is equal to the less EHF.



Take K, L, the centres of the circles, and draw BK, KC, and EL, LF. Because the circles are equal, their radii are equal; therefore BK, KC are equal to EL, LF. But the base BC is also equal to EF; therefore the angle BKC is equal to ELF; therefore the arch BGC is equal to the arch EHF (26. 3). But the circumferences ABGC, DEHG are equal; therefore the remaining arches BAC, EDF are equal. Therefore, in equal circles &c. Q. E. D.

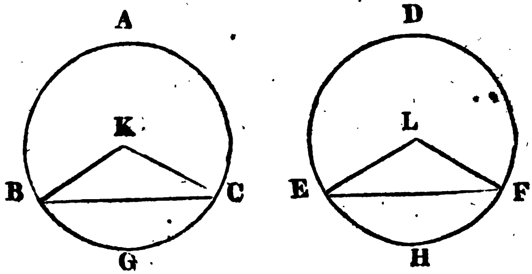
Cor. In the same circle equal chords cut off equal arches.

Scholium. This prop. is sometimes expressed thus. In equal circles equal chords subtend equal arches, the greater arch equal to the greater, and the less arch equal to the less.

PROPOSITION XXIX. THEOREM.

In equal circles equal arches are subtended by equal chords.

Let ABC, DEF be equal circles, and BGC, EHF equal arches; join BC, EF; the chords BC and EF are equal.



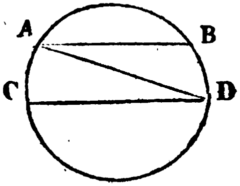
Take K, L, the centres of the circles, and draw BK, KC, and EL, LF. Because the arch BGC is equal to EHF, the angle BKC is equal to ELF (27. 3); and because the circles ABC, DEF are equal, their radii are equal; therefore BK, KC are equal to EL, LF; and they contain equal angles; therefore the base BC is equal to EF. Therefore, in equal circles &c. Q. E. D.

Cor. In the same circle equal arches are subtended by equal chords.

PROPOSITION A. THEOREM. Ed.

Parallel chords in a circle intercept equal arches.

Let the chords AB, CD be parallel; the arches AC, BD, which they intercept, are equal.



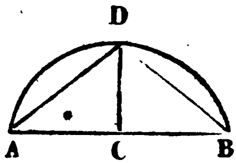
Draw AD. Because the straight lines AB, CD are parallel, the alternate angles BAD, ADC are equal (29. 1), therefore the arches AC, BD, on which those angles stand, are equal (26. 3). Therefore, parallel chords &c. Q. E. D. Ed.

PROPOSITION XXX. PROBLEM.

To bisect a given arch.

Let ADB be the given arch; it is required to bisect it.

Draw AB, and bisect it in C; from the point C draw CD perp. to AB, and join AD, DB; the arch ADB is bisected in the point D.



AC is equal to CB, and CD is common to the triangles ACD, BCD, and the right angles at C are equal; therefore the base AD is equal to BD; wherefore the arch AD is equal to the arch DB (28. 3); and each of the arches AD, DB is less than a semicircle, because DC passes through the centre (Cor. 1. 3). Therefore the given arch ADB is bisected in D. Which was to be done.

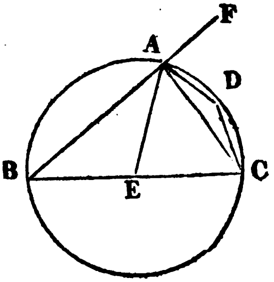
Cor. Hence, if a straight line bisect a chord at right angles, it will also bisect the arch subtended by that chord. This is evident from the demonstration. Ed.

PROPOSITION XXXI. THEOREM.

The angle at the circumference of a semicircle is a right angle; the angle in a segment greater than a semicircle is acute; and the angle in a segment less than a semicircle is obtuse.

Let ABCD be a circle, of which the diameter is BC, and the centre E; draw CA dividing the circle into the two segments ABC, ADC; draw BA, AD, DC; the angle in the semicircle BAC is a right angle; the angle in the segment ABC, which is greater than a semicircle, is acute; and the angle in the segment ADC, which is less than a semicircle, is obtuse.

Join AE, and produce BA to F. Because BE is equal to EA, the angle EAB is equal to EBA; and because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB. But FAC, the exterior angle of the triangle ABC, is also equal to the two angles ABC, ACB (32. 1); therefore the angle BAC is equal to FAC; therefore each of them is a right angle. Wherefore the angle BAC in a semicircle is a right angle.



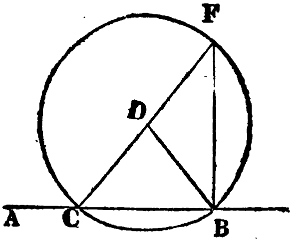
Again, the angle BAC of the triangle ABC is a right angle, therefore ABC is less than a right angle (3 Cor 32. 1). Therefore the angle in a segment ABC, greater than a semicircle, is acute.

Lastly, because ABCD is a quadrilateral figure in a circle, the angles ABC, ADC are together equal to two right angles (22. 3). But ABC is an acute angle; wherefore ADC is an obtuse angle. Therefore the angle in a segment ADC, less than a semicircle, is obtuse. Therefore, the angle &c. Q. E. D.

Cor. A straight line AE drawn from the right angle BAC of a right angled triangle ABC to the middle E of the hypothenuse BC is equal to half the hypothenuse. For since the angle BAC is in a semicircle AE is equal to BE. Ed.

Scholium. From the remarkable property of a circle, that the angle in a semicircle is a right angle, are derived elegant methods of drawing tangents to circles, and perpendiculars from any points in or without a straight line.

1. To erect a perpendicular at any point B in a given straight line AB.



Without the line AB take any point D which is opposite to AB, and draw DB. From the centre D, with the radius DB, describe a circle cutting AB in the points C and B. Through the points C and D draw the line CDF meeting the circumference in F. Draw BF, which will be the perpendicular required.

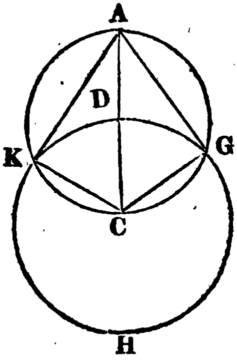
For the angle CBF is in a semicircle, and therefore is a right angle; therefore BF is perp. to AB.

2. To let fall a perpendicular on a given straight line AB from a given point F without it.

From F draw any inclining line FC to meet AB in any point C; bisect FC in D, and from the point D, with the radius DC, describe a circle cutting AB in C and B. Draw FB, which is the perpendicular required.

For the angle CBF is in a semicircle, and therefore is a right angle; therefore FB is perp. to AB.

3. To draw a tangent to a circle from a given point without it.



Let DGH be the given circle, and A any point without it, from which it is required to draw a tangent.

Find the centre C, and join AC, and on AC as a diameter describe the circle AGCK cutting the circumference of the given circle in the points G and K. Draw AG, AK, then AG or AK is the tangent required.

Draw CG and CK, then the angles CGA, CKA in semicircles are right angles, therefore AG, AK are perp. to the radii CG, CK, and therefore touch the circle DGH in the points G and K (Cor. 16. 3).

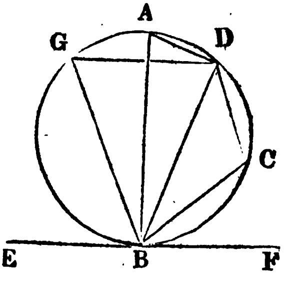
Cor. Hence two tangents AG, AK, drawn from the same point A without a circle, are equal. For the triangles AGC, AKC, having the side CG equal to CK, and CA common, and the angles at G and K right, are equal in all respects (3 Cor. B. 2), therefore AG is equal to AK.

PROPOSITION XXXII. THEOREM.

If a straight line touch a circle, and a chord be drawn from the point of contact, the angle made by the tangent and chord is equal to the angle in the segment of the circle, which is on the opposite side of the chord.

Let the line EF touch the circle ABCD in the point B, and from B let the chord BD be drawn; the angle FBD which BD makes with BF is equal to the angle which is the opposite segment DAGB, and the angle DBE, which BD makes with BE, is equal to the angle in the opposite segment BCD.

From the point B draw BA perp. to EF; take any point C in the arch BD, and draw AD, DC, CB. Because EF touches the circle ABCD in B, and BA is drawn perp. to EF form the point of contact B, the centre of the circle is in BA (19. 3), and BA is a diameter; therefore ADB is a right angle (31. 3), and therefore the other two angles BAD, ABD are together equal to a right angle (4 Cor. 32. 1). But ABF is a right angle; therefore ABF is equal to the two angles BAD, ABD. Take from these equals the common angle ABD, and there will remain the angle DBF equal to the angle BAD which is in the opposite segment of the circle, and is equal to any other angle BGD in the same segment (21. 3).

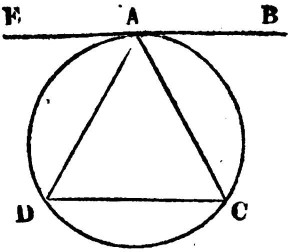


Again, the three angles EBD, DBC, CBF, are together equal to two right angles (1 Cor. 13. 1), and the three angles of the triangle BDC are together equal to two right angles (32. 1); therefore EBD + DBC + CBF = BDC + DBC + BCD. But in these two equals the angle DBC is common, and the angle CBF is equal to BDC, by the former case; therefore the angle EBD is equal to the angle BCD, which is in the opposite segment of the circle, and is equal to any angle in that segment. Wherefore, if a straight line &c. Q. E. D. ED.

Cor. If a straight line FB meet the circumference of a circle in any point B, and make an angle with a chord BD drawn from the same point B, equal to the angle at A in the alternate segment BGD of the circle, the line BF touches the circle in that point. Ed.

This proposition may be thus demonstrated. Ed.

Let the straight line EB touch the circle ACD in the point A, and from A let the chord AC be drawn. Through C draw CD parallel to EB, and join AD. Then the angle BAC is equal to ACD (29. 1); and because the arches AC, AD are equal (A. 3), the angles ACD, ADC are equal (27. 3). Therefore the angle BAC is equal to the angle ADC in the alternate segment ADC.

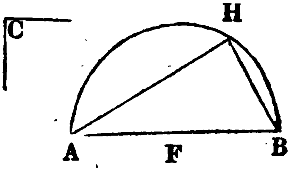


PROPOSITION XXXIII. PROBLEM.

On a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle; that is, to describe a segment such that all the angles described in it shall be equal to a given angle.

Let AB be the given straight line, and C the given rectilineal angle; it is required to describe on AB a segment of a circle containing an angle equal to C.

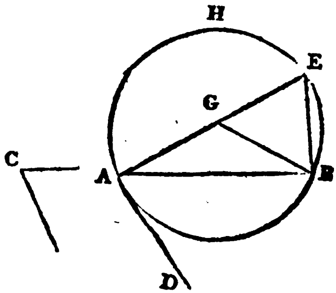
First, let C be a right angle; bisect AB in F, and from the centre F, at the distance FB, describe the semicircle AHB. The angle AHB in a semicircle is equal to the right angle C (31 3).

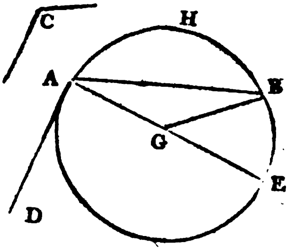


If C be not a right angle, then at the point A, in the line AB, make the angle BAD equal to C (23. 1), and from A draw AE perp. to AD; at the point B, the other extremity of AB, make the angle ABG equal to BAG; then GA is equal to GB (6. 1). From the centre G, at the distance GA, describe a circle, and it

will pass through the point B; then AHB is the segment required.

Because AD is perp. to the extremity of the diameter AE, it touches the circle in the point A (Cor. 16. 3); and since AB is a chord drawn from the point of contact A, the angle BAD is equal to any angle in the segment on the other side of AB (32. 3). But the angle BAD is equal to C; therefore the angle C is equal to any angle in the segment AHB on the other side of AB. Wherefore, on the given line AB the segment AHB of a circle is described which contains an angle equal to the given angle C. Which was to be done.



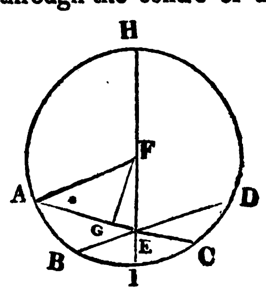


PROPOSITION XXXV. THEOREM.*

If two chords of a circle cut each other, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other.

Let the two chords AC, BD, of the circle ABCD, cut each other in the point E; the rectangle AE . EC = BE . ED.

Let the chord HI which passes through the centre of the circle, cut the chord AC, which does not pass through the centre, in E; bisect HI in F, then F is the centre of the circle. Join AF, and from F draw FG perp. to AC; then AG is equal to GC (3. 3). In the triangle AEF the sum of the sides AF, FE is HE, and their difference is EI; also, the sum of the segments of the base AG, GE, made by the perp. FG, is AE, and their difference is AG — GE = GC — GE = EC.



* Shorter and plainer demonstrations of this and the next proposition may be seen at the end of Book VI.

But $(AF + FE).(AF - FE) = (AG + GE).(AG - GE)$ (F. 2), that is, $HE.EI = AE.EC$.

Again, if any chord HI, which passes through the centre, cut any other chord BD, which does not pass through the centre, it may be proved in the same manner that $BE.ED = HE.EI$. And it has been proved that $AE.EC = HE.EI$. Consequently $AE.EC = BE.ED$. Therefore, if two chords &c. Q. E. D. Ed.

Cor. 1. If two chords of a circle cut each other, their segments are reciprocally proportional. For $AE.EC = BE.ED$, $\therefore AE : ED :: EB : EC$.

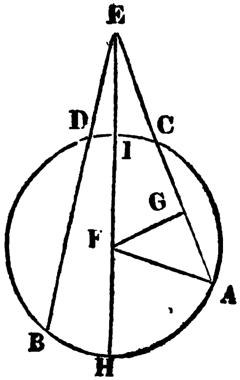
Cor. 2. If from any point C (figure, Prop. C), in the circumference of a circle a perpendicular CE be drawn to a diameter AB, the rectangle contained by the segments of the diameter is equal to the square of the perpendicular. Produce CE to the circumference at D, then $CE = ED$ (3. 3), and $AE.EB = CE.ED$ or CE^2.

PROPOSITION B. THEOREM.

This is Cor. 36. 3.

If from any point without a circle two secants be drawn, the rectangle contained by one secant and its external part is equal to the rectangle contained by the other secant and its external part.

Let E be any point without the circle ACDB, from which two secants EA, EB are drawn; the rectangle $AE.EC = BE.ED$.



Through the point E and the centre F draw the secant EFH; join AF, and from F draw FG perp. to AC; then $AG = GC$ (3. 3). In the triangle AEF the sum of the sides AF, FE is HE, and their difference is IE; also the sum of the segments of the base AG, GE, made by the perp. FG, is AE, and their difference is CE. But $(FE + FA).(FE - FA) = (GE + GA).(GE - GA)$ (F. 2), that is, $HE.EI = AE.EC$. In the same manner it may be proved that $HE.EI = BE.ED$. Consequently $AE.EC = BE.ED$. Therefore, if from any &c. Q. E. D. Ed.

Cor. If from any point without a circle two secants be drawn, they are to each other reciprocally as their external segments. For $AE.EC = BE.ED$, $\therefore AE : BE :: ED : EC$.

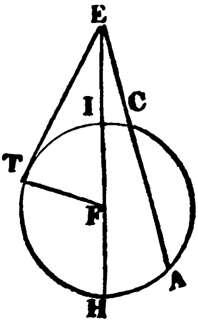
Scholium. The last two propositions may be enunciated together as follows.

If two chords intersect each other in a circle, or, being produced, without it, the rectangle of the segments of one chord, intercepted between the point of concourse and the circumference, is equal to the rectangle of the similar segments of the other.

PROPOSITION XXXVI. THEOREM.

If from any point without a circle a tangent and a secant be drawn, the rectangle contained by the secant and its external part is equal to the square of the tangent.

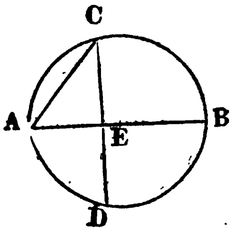
Let E be any point without a circle ACTH, from which a tangent ET and a secant EA are drawn; $AE.EC = ET^2$.



Let EH pass through the centre F, and join FT; then FTE is a right angle (18.3), therefore $FE^2 - FT^2 = TE^2$ (1 Cor. B. 2). But $FE^2 - FT^2 = (FE + FT).(FE - FT)$ (5.2), or $HE.EI = ET^2$. But $HE.EI$ is equal to the rectangle $AE.EC$ under any other secant and its external part (B. 3). Therefore $AE.EC = ET^2$. Therefore, if from any &c. Q. E. D.

PROPOSITION C. THEOREM. Ed.

If from any point in the circumference of a circle a perpendicular be drawn to a diameter, and a chord to either end of it, the square of the chord is equal to the rectangle under the diameter and the part of it between the perpendicular and the chord.



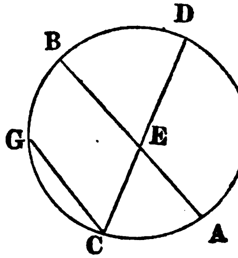
From any point C in the circumference of the circle ACBD let a perp. CE and a chord CA be drawn to the diameter AB; $AC^2 = AB.AE$.

For $AC^2 = CE^2 + AE^2$ (B. 2), and $AB.AE = AE^2 + AE.EB$ (3. 2). But $CE^2 = AE.EB$ (2 Cor. 35. 3); therefore $AB.AE = AE^2 + CE^2$, or AC^2. Therefore, if from any point &c. Q. E. D.

PROPOSITION D. THEOREM. Ed.

If two chords of a circle intersect each other, the angle which they contain is equal to half the angle at the centre which stands on the sum or the difference of the arches intercepted between them, according as they meet within or without the circle.

1. Let ACBD be a circle, and AB, CD two chords which intersect each other in the point E within the circle; the angle DEB is equal to half the angle at the centre which stands on an arch equal to the sum of the arches AC, DB.

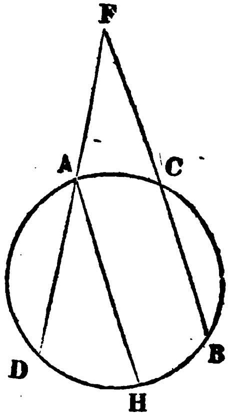


Through C draw CG parallel to AB, then the arch BG = AC (A. 3), and therefore the arch DBG = AC + DB. Now the angle DEB = angle DCG (29. 1), which is equal to half the angle at the centre standing on the same arch DBG (20. 3); therefore the angle DEB is equal to half the angle at the centre standing on the arch DBG.

2. Again, let the two chords DA, BC be produced and meet in F; the angle DFB is equal to half the angle at the centre standing on an arch equal to the difference of the arches DB, AC.

Draw AH parallel to BC, then the arch HB = AC, and the arch DH = DB — AC.

Now the angle DFB = angle DAH, which is equal to half the angle at the centre standing on the arch DH; therefore the angle DFB is equal to half the angle at the centre standing on the arch DH. Therefore, if two chords &c. Q. E. D.

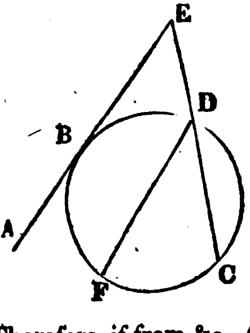


Cor. If two chords intersect each other at right angles within a circle, the sum of the opposite arches which they intercept is equal to half the circumference.

PROPOSITION E. THEOREM. Ed.

If from any point without a circle a tangent and a secant be drawn, the angle which they contain is equal to an angle at the circumference of the circle standing on an arch equal to the difference between the arches intercepted by the tangent and secant.

From any point E without the circle BDC let a tangent EBA and a secant EDC be drawn; the angle AEC is equal to an angle at the circumference on an arch equal to the difference of the arches BC, BD.



From the point D draw DF parallel to AE, then the arch FB is equal to BD (A. 3), therefore the arch FC is equal to the difference of the arches BC and BD.

Again, because FD is parallel to AE, the angle FDC is equal to AEC (29. 1). But FDC is an angle at the circumference standing on the arch FC, which is equal to the difference of the arches BC, BD. Therefore, if from &c. Q. E. D.

THE PRINCIPAL THEOREMS IN BOOK III.

If in a circle a straight line bisect a chord at right angles, it will pass through the centre of the circle.

If in a circle a diameter bisect a chord, it will be perpendicular to the chord; and if it be perpendicular to a chord, it will bisect the chord, and also the arch subtended by the chord.

Chords in a circle, which are equal to one another, are equally distant from the centre; and chords which are equally distant from the centre of a circle are equal to one another.

A straight line perpendicular to the extremity of a radius or a diameter is a tangent to the circle.

If a straight line touch a circle, a radius drawn to the point of contact is perpendicular to the tangent.

An angle at the centre of a circle is double of an angle at the circumference, standing on the same arch.

All angles in the same segment of a circle, or standing on the same arch, are equal to one another.

The two opposite angles of any quadrilateral figure described in a circle are together equal to two right angles.

In equal circles equal angles stand on equal arches, whether they be at the centres or at the circumferences of the circles.

In equal circles the angles which stand on equal arches are equal to one another, whether they be at the centres or at the circumferences of the circles.

In equal circles equal chords subtend equal arches on the same sides of them, and equal arches are subtended by equal chords.

An angle in a semicircle is a right angle.

If a straight line touch a circle, and a chord be drawn from the point of contact, the angle made by the tangent and chord is equal to the angle in the segment of the circle which is on the other side of the chord.

If two chords in a circle cut each other, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other.

If from any point without a circle two secants be drawn, the rectangle contained by one of them and its external part is equal to the rectangle contained by the other and its external part.

If from any point without a circle a tangent and a secant be drawn, the rectangle contained by the secant and its external part is equal to the square of the tangent.

If from any point in the circumference of a circle a perpendicular be drawn to a diameter, and a chord to either end of it, the square of the perpendicular is equal to the rectangle under the segments of the diameter; and the square of the chord is equal to the rectangle of the diameter and its segment adjacent to the chord.

If two chords of a circle intersect each other, the angle which they contain is equal to half the angle at the centre which stands on the sum or the difference of the arches intercepted between them, according as they meet within or without the circle.

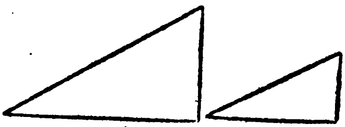
END OF BOOK III.

ELEMENTS OF GEOMETRY.

BOOK VI.

DEFINITIONS.

1. Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportional.



A. In similar figures the sides which are opposite to equal angles are homologous or like sides. Ed.

2. Two sides of one figure are said to be reciprocally proportional to two sides of another, when one of the sides of the first figure is to one of the sides of the other, as the remaining side of the other is to the remaining side of the first.

B. Two magnitudes, and other two, are said to be reciprocally proportional, when one magnitude of the former is to one of the latter, as the remaining magnitude of the latter is to the remaining magnitude of the former. Ed.

C. A straight line is said to be harmonically divided, when it is divided into three segments, so that the whole line is to one of the extreme segments as the other extreme segment is to the middle segment. Ed.

D. Two straight lines are said to be similarly divided, when any two corresponding segments of them have the same ratio as the lines have. Ed.

3. A straight line is said to be cut in extreme and mean ratio, when the whole line is to the greater segment as the greater segment is to the less.

4. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

PROPOSITION I. THEOREM.

Triangles of the same altitude are to one another as their bases; and parallelograms of the same altitude are to one another as their bases.*—*See Note.*

Let P and p denote two parallelograms, A and a their altitudes, B and b their bases; then, by Preliminary Observations to Book II, the surface of $P = A.B$, and the surface of $p = a.b$. Consequently $P : p :: A.B : a.b$; that is, parallelograms are to one another as their bases and altitudes jointly.

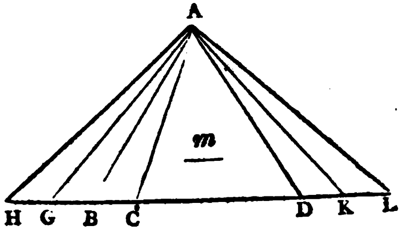
Because a paral. is double of a triangle of the same base and altitude (41. 1), triangles are to one another as their bases and altitudes jointly.

If the altitudes of the parallelograms or triangles be equal, the proportion becomes $P : p :: B : b$, that is, parallelograms or triangles of the same altitude are to one another as their bases, which is the proposition.

Cor. If the bases be equal, then $P : p :: A : a$, that is, parallelograms or triangles on equal bases are to one another as their altitudes.

Otherwise. Let the triangles ACH and ADL have the same altitude; the triangle ACH : ADL : : HC : DL.

Let the bases HC and DL be commensurable, or be divisible by the same quantity, and let the line m be their common measure or divisor. Divide the base HC into the parts CB, BG, GH, each equal to m; and divide the base DL into the parts DK, KL, each equal to m. From the vertex A draw straight lines to the several points of division.



The triangles ABC, AGB, AHG, ADK, AKL, are equal to one another (38. 1). The triangle ACH contains ABC as often as the base HC contains m, and the triangle ADL contains ADK, or its equal ABC, as often as the base DL contains m. Therefore the triangle ACH : ABC : : base HC : m (Propor. 19), and the triangle ADL : ABC : : base DL : m. These

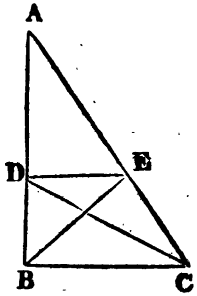
* Euclid's demonstration of this proposition is difficult to students, and therefore has been changed for another which is simpler and more intelligible.

two proportions become by alternation (Propor. 36), the triangle ACH : base HC : : ABC : *m*, and the triangle ADL : base DL : : ABC : *m*. Hence ACH : HC : : ADL : DL (Propor. 34), or the triangle ACH : ADL : : base HC : DL.

Again, because a paral. is double of a triangle of the same base and altitude (41. 1), 2ACH : 2ADL : : HC : DL (Propor. 30). Therefore, triangles &c. Q. E. D. ED.

PROPOSITION II. THEOREM.

If a straight line be drawn parallel to one of the sides of a triangle it will cut the other sides proportionally.



Let DE be drawn parallel to BC, one of the sides of the triangle ABC; BD is to DA as CE to EA.

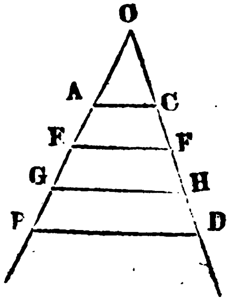
Join BE, CD; then the triangle BDE is equal to the triangle CDE (37. 1). But ADE is another triangle; therefore the triangle BDE : triangle ADE : : triangle CDE : triangle ADE. But the triangle BDE : triangle ADE : : BD : DA (1. 6), and the triangle CDE : triangle ADE : : CE : EA. Therefore BD : DA : : CE : EA (Propor. 34). Therefore, if a straight line &c. Q. E. D.

Cor. BD + DA : DA : : CE + EA : EA, or AB : AD : : AC : AE (Propor. 37). Also, AB : BD : : AC : CE.

PROPOSITION A. THEOREM. LEGENDRE.

If between two straight lines any number of parallel straight lines be drawn, those two lines will be cut proportionally.

Let AB, CD be two straight lines, and let any number of parallels AC, EF, GH, BD, &c. be drawn between them; AE : CF : : EG : FH : : GB : HD.



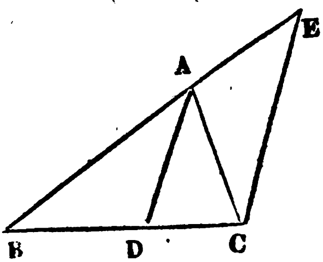
Produce the lines AB, CD to meet in O. In the triangle OEF, where AC is parallel to the base EF, OE : AE : : OF : CF (Cor. 2. 6), ∴ OE : OF : : AE : CF (Propor. 36). In the triangle OGH we have OE : EG : : OF : FH, ∴ OE : OF : : EG : FH. Consequently AE : CF : : EG : FH (Propor. 34). In the same manner EG : FH : : GB : HD; and so on, as far as we please. Therefore the two lines AB, CD are cut proportionally by the parallels EF, GH, &c. Q. E. D.

PROPOSITION III. THEOREM.

If an angle of a triangle be bisected by a straight line which cuts the base or opposite side, the segments of the base will have the same ratio which the adjacent sides of the triangle have to each other; and if the segments of the base have the same ratio which the adjacent sides of the triangle have to each other, the straight line drawn from the vertex to the point of section bisects the vertical angle.

Let the angle BAC, of any triangle ABC, be bisected by the straight line AD; BD is to DC as BA to AC.

Through the point C draw CE parallel to DA, and let BA produced meet CE in E. Because the line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD (29. 1). But CAD, by hypothesis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE.



Again, because the line BAE meets the parallels AD, EC, the exterior angle BAD is equal to the interior angle AEC. But the angle ACE has been proved equal to BAD; therefore also ACE is equal to the angle AEC; consequently the side AE is equal to AC. Because AD is drawn parallel to EC, one of the sides of the triangle BCE, BD is to DC as BA to AE (2. 6) or AC.

Next, let BD be to DC as BA to AC, and join AD; the angle BAC is bisected by the straight line AD.

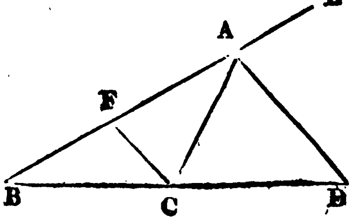
The same construction being made, as BD to DC so is BA to AC; and as BD to DC so is BA to AE (2. 6), because AD is parallel to EC; therefore AB is to AC as AB to AE (Propor. 34); consequently AC is equal to AE (Propor. 32), therefore the angle AEC is equal to the angle ACE. But the angle AEC is equal to the exterior and opposite angle BAD, and the angle ACE is equal to the alternate angle CAD (29. 1); wherefore also the angle BAD is equal to the angle CAD; therefore, the angle BAC is divided into two equal angles by the straight line AD. Therefore, if the angle &c. Q. E. D.

COR. If any angle of a triangle be bisected by a straight line which meets the opposite side, the sum of the two sides containing that angle is to either of them, as the base is to the segment adjacent to that side. For, since BA : AC : : BD : DC, BA + AC : BA : : BD + DC or BC : BD (Propor. 37). ED.

PROPOSITION B. THEOREM.

If one side of a triangle be produced, and the exterior angle be bisected by a straight line which cuts the base produced, the segments between the bisecting line and the extremities of the base have the same ratio which the adjacent sides of the triangle have to each other; and if the segments of the base produced have the same ratio which the adjacent sides of the triangle have, the straight line drawn from the vertex to the point of section bisects the exterior angle of the triangle.

Let the exterior angle CAE, of any triangle ABC, be bisected by the straight line AD, which meets the base produced in D; BD is to DC as BA to AC.



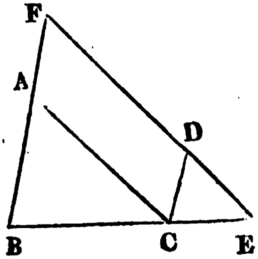
Through C draw CF parallel to AD. The proof is literally the same as that of the last proposition, and need not be repeated.

PROPOSITION IV. THEOREM.

The sides about the equal angles of equiangular triangles are proportional; and the sides which are opposite to the equal angles are homologous sides, that is, they are the antecedents or the consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and consequently the angle BAC equal to the angle CDE (2 Cor. 32. 1). The sides about the equal angles of the triangles ABC, DCE are proportional; and those are the homologous sides which are opposite to the equal angles. "Thus, the angle at A being equal to the angle at D, the sides about the former are proportional to the sides about the latter, namely, AB : AC : : DC : DE; also the two antecedents AB, DC are opposite to the equal angles at C, E; and the two consequents AC, DE are opposite to the equal angles at B, C."

Let the triangle DCE be placed so that its side CE may be contiguous to BC, and in the same straight line with it. Because the angles ABC, ACB are together less than two right angles, the angles ABC, DEC are also less than two right angles; wherefore BA, ED produced will meet. Let them be produced, and meet in the point F. Because the angle ABC is equal to DCE, BF is parallel to CD (28. 1); and because the angle ACB is equal to DEC, AC is parallel to FE. Therefore FACD is a parallelogram; consequently AF is equal to CD (34. 1), and AC to FD.



Because AC is parallel to FE, one of the sides of the triangle FBE, BA : AF : : BC : CE (2. 6), or, because AF is equal to CD, BA : CD : : BC : CE; therefore, alternately, BA : BC : : DC : CE (Propor. 36). Again, because CD is parallel to BF, BC : CE : : FD : DE, or, because FD is equal to AC, BC : CE : : AC : DE;

therefore BC : CA : : CE : ED.

But BA : BC : : DC : CE;

therefore BA : AC : : CD : DE (Propor. 42, 43). Therefore, the sides &c. Q. E. D.

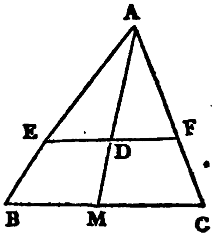
Cor. It follows from this prop. that equiangular triangles are of necessity *similar* figures, according to Def. 1; therefore

similar triangles are often used to denote equiangular triangles. LUDLAM.

PROPOSITION C. THEOREM. LUDLAM.

If a straight line be drawn from the vertex of any triangle to the base, it will cut every line which is parallel to the base in the same ratio as the segments of the base.

Let ABC be a triangle, BC the base, EF a line parallel to the base, ADM a line drawn from the vertex A, cutting EF in D, and the base in M; the segments of EF have the same ratio to each other as the segments of the base BC; DE : DF : : MB : MC.



In the triangles AED, ABM the angles AED, ABM are equal (29. 1), and the angle FAD is common, therefore the remaining angles ADE, AMB are equal (2 Cor. 32. 1); therefore the triangles AED, ABM are equiangular. In the same manner the triangles ADF, AMC are equiangular. Hence

AD : AM : : DE : MB (4. 6),
and AD : AM : : DF : MC,
therefore DE : MB : : DF : MC (Propor. 34);
therefore DE : DF : : MB : MC (Propor. 36).

COR. 1. If a straight line drawn from the vertex of a triangle to the base bisect the base, it will bisect every line parallel to the base, and terminating in the two sides of the triangle.

COR. 2. Parallel lines BC, EF are cut proportionally by diverging lines AB, AM and AC, AM. For MB : MC : : DE : DF.

COR. 3. Diverging lines are proportional to the corresponding segments into which they divide parallel lines; AB : AE : : BM : ED (4. 6) : : MC : DF.

PROPOSITION D. THEOREM. ED.

If a straight line be drawn parallel to the base of a triangle, and another straight line be drawn from the vertex to meet the two parallels, the segments of that line intercepted between the vertex and the parallels will be to each other as the two parallels.—*See last fig.*

Let a straight line EF be drawn parallel to the base BC of a triangle ABC, and from the vertex A let AM be drawn to meet the parallels BC, EF; AM : AD : : BC : EF.

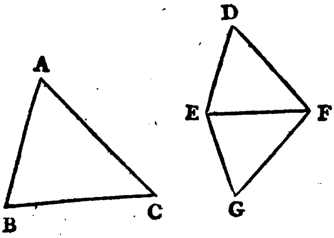
For the triangles AEF, ABC are equiangular, and also the triangles AED, ABM;

therefore AE : EF : : AB : BC (4. 6),
and AD : AE : : AM : AB;
therefore AD : EF : : AM : BC (Propor. 42, 43),
∴ AD : AM : : EF : BC (Propor. 36).

PROPOSITION V. THEOREM.

If the sides of two triangles, about each of their angles, be proportional, the triangles will be equiangular, and will have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportional, so that AB : BC : : DE : EF,
and BC : CA : : EF : FD,
and BA : AC : : ED : DF;
the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides, namely, the angle ABC being equal to the angle DEF, and the angle BCA to EFD, and also the angle BAC to EDF.



At the points E, F, in the line EF, make the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the angle BAC is equal to EGF (2 Cor. 32. 1); therefore the triangle ABC is equiangular to the triangle GEF; consequently they have their sides opposite to the equal angles proportional (4. 6). Wherefore

AB : BC : : GE : EF. But, by supposition,
AB : BC : : DE : EF; therefore
DE : EF : : GE : EF (Propor. 34).

Therefore DE and GE are equal (Propor. 32). For the same reason DF is equal to FG,

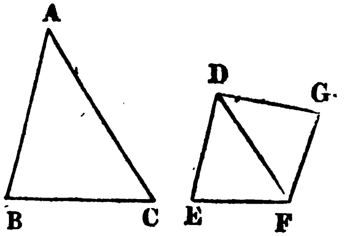
In the triangles DEF, GEF, DE is equal to EG, and EF common, and DF equal to GF; therefore the angle DEF is equal to GEF; wherefore the angle DFE is equal to GFE, and the angle EDF to EGF. Now the angle DEF is equal to GEF,

and GEF to the angle ABC; therefore the angle ABC is equal to DEF. For the same reason the angle ACB is equal to DFE, and the angle A to the angle D. Therefore the triangle ABC is equiangular to the triangle DEF. 'And they have their equal angles opposite to the homologous sides, namely, the equal angles ACB and DFE opposite to the homologous sides AB and DE, and the equal angles A and D opposite to the homologous sides BC and EF, and the equal angles ABC and DEF opposite to the homologous sides AC and DF.' Wherefore, if the sides &c. Q. E. D.

PROPOSITION VI. THEOREM.

If two triangles have one angle of one triangle equal to one angle of the other, and the sides about the equal angles proportional, the triangles will be equiangular, and will have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF have the angle BAC in one triangle equal to the angle EDF in the other, and the sides about those angles proportional, that is, BA to AC as ED to DF; the triangles ABC, DEF are equiangular, and have those angles equal which are opposite to the homologous sides, namely, the angle ABC equal to the angle DEF, and the angle ACB equal to DFE.



At the points D, F, in the line DF, make the angle FDG equal to either of the angles BAC, EDF, and the angle DFG equal to ACB; wherefore the remaining angle B is equal to G (2 Cor. 32. 1); consequently the triangle ABC is equiangular to the triangle DGF; therefore BA : AC : : GD : DF (4. 6). But, by hypothesis,

BA : AC : : ED : DF; therefore

ED : DF : : GD : DF (Propor. 34); wherefore ED is equal to DG (Propor. 32).

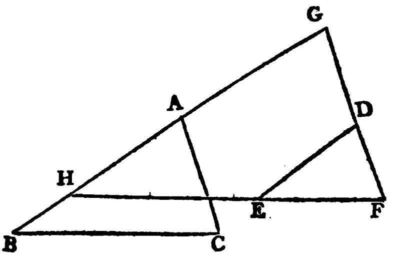
The triangles EDF, GDF have the two sides ED, DF equal to the two sides GD, DF, and also the angle EDF equal to GDF; wherefore the base EF is equal to FG, and the angle

DFG is equal to DFE, and the angle G to the angle E. But the angle DFG is equal to ACB; therefore the angle ACB is equal to DFE; also the angle BAC is equal to EDF; wherefore the remaining angle B is equal to E. Therefore the triangle ABC is equiangular to the triangle DEF. 'And they have the angles ABC, DEF equal, which are opposite to the homologous sides AC, DF, and the angles ACB, DFE equal, which are opposite to the homologous sides AB, DE.' Wherefore, if two triangles &c. Q. E. D.*

PROPOSITION E. THEOREM. LEGENDRE.

If the homologous sides of two triangles be parallel to each other, or perpendicular each to each; the triangles are similar.

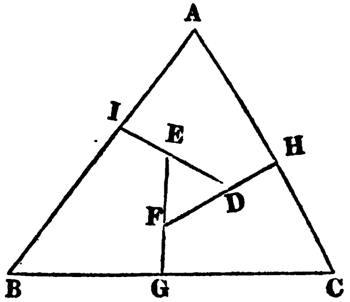
1. Let ABC, DEF be two triangles which have the sides AB, BC, AC, parallel to the sides DE, EF, DF, each to each; the triangles ABC and DEF are similar.



Because AB is parallel to DE, and BC to EF, the angle ABC is equal to DEF (D. 1); and because AC is parallel to DF, the angle ACB is equal to DFE; consequently the angle BAC is equal to EDF. Hence the triangles ABC, DEF are equiangular.

* See a more simple demonstration in Hutton's Mathematics, Geometry, Theor. 86; or West's Mathematics, Prop. V. B. V. ED.

2. Let ABC, DEF be two triangles which have the sides DE, DF, EF, perpendicular to the sides AB, AC, BC, each to each; the triangles ABC and DEF are similar.



In the quadrilateral figure AIDH the two angles at I and H are right, and the four angles are together equal to four right angles (6 Cor. 32. 1); therefore the two remaining angles IAH, IDH are together equal to two right angles. But the two angles EDF, IDH are together equal to two right angles (13. 1); therefore the angle EDF is equal to IAH or BAC.

If the side EF be perp. to BC it may be proved in the same manner that the angle DFE is equal to C, and the angle DEF to B. Hence the triangles ABC, DEF, whose sides are perp. each to each, are equiangular and similar.

Therefore, if the homologous sides &c. Q.E.D.

Scholium. In the case of parallel sides the homologous sides are the parallel sides; and in the case of perpendicular sides the homologous sides are the perpendicular sides. Thus, in case 2, DE is homologous to AB, DF to AC, and EF to BC.

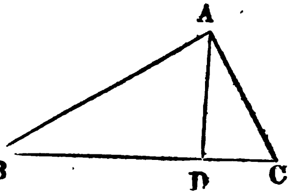
PROPOSITION VIII. THEOREM.

In a right angled triangle, if a perpendicular be drawn from the right angle to the hypothenuse, the two triangles on each side of it are similar to the whole triangle and to each other.

Let ABC be a right angled triangle, having the right angle BAC, and from the point A let AD be drawn perp. to the hypothenuse BC; the triangles ABD, ADC are similar to the whole triangle ABC, and to each other.

Because the angle BAC is equal to ADB, each of them being a right angle, and the angle B is common to the two triangles ABC, ABD, the remaining angle ACB is equal to BAD (2 Cor. 32. 1); therefore the triangle ABC is equiangular to

the triangle ABD; therefore the sides about the equal angles are proportional (4. 6); wherefore the triangles are similar (1 Def. 6). In like manner it may be proved that the triangle ADC is equiangular and similar to the triangle ABC.



Because the angles at D are right angles, and the angles ACD, DAB have been proved to be equal, the remaining angles ABD, DAC are equal; therefore the triangles ABD, ACD are mutually equiangular; consequently their homologous sides are proportional (4. 6); wherefore the triangles are similar. Therefore, in a right angled &c. Q. E. D.

Cor. 1. The perpendicular drawn from the right angle of a right angled triangle to the hypothenuse or base, is a mean proportional between the segments of the base; and each of the sides is a mean proportional between the base and its segment adjacent to that side.

For, in the equiangular triangles
BDA, ADC, BD : DA : : DA : DC;
and in the triangles ABC, DBA, BC : BA : : BA : BD;
and in the triangles ABC, ACD, BC : CA : : CA : CD.

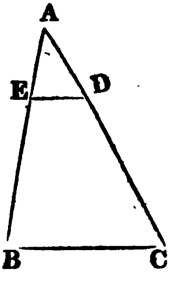
Cor. 2. Hence $AB^2 = BC \cdot BD$ (Propor. 26),
and $AC^2 = BC \cdot DC$; therefore,
by adding, $AB^2 + AC^2 = BC \cdot BD + BC \cdot DC$, or $BC \cdot BC$ (3. 2), or BC^2. *This is Prop. B.* 2. Ed.

PROPOSITION IX. PROBLEM.

To divide a given straight line into two parts which shall be in any given ratio.

Let AB be the given straight line; it is required to divide AB into two parts which shall be in a given ratio.

From the point A draw a line AC, making any angle with AB; in AC take any point D, and let AD : DC be the given ratio. Join BC, and draw DE parallel to BC; then AB is divided in E in the ratio of AD to DC.



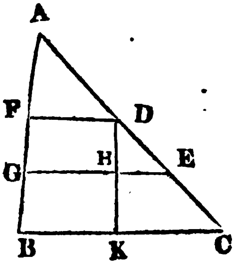
Because ED is parallel to BC, one of the sides of the triangle ABC, CD : DA : : BE : EA (2. 6), or AD : DC : : AE : EB (Propor. 35). Therefore the given line AB is divided into two parts which are in a given ratio. Which was to be done.

PROPOSITION X. PROBLEM.

To divide a given straight line similarly to a given divided straight line, that is, into parts which shall have the same ratios to one another which the parts of the divided given line have to one another.

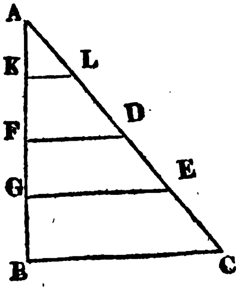
Let AB be the straight line given to be divided, and AC the divided line; it is required to divide AB similarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed so as to contain any angle; join BC, and through the points D, E draw DF, EG parallel to BC, and through D draw DHK parallel to AB; then each of the figures FH, HB is a paral.; wherefore DH is equal to FG (34. 1), and HK to GB.



Because HE is parallel to KC, one of the sides of the triangle DKC, CE : ED : : KH : HD (2. 6). But KH = BG, and HD = GF; therefore CE : ED : : BG : GF. Again, because FD is parallel to EG, one of the sides of the triangle AGE, ED : DA : : GF : FA. Therefore the given line AB is divided similarly to AC. Which was to be done.

Cor. Hence a given straight line may be divided into any proposed number of equal parts.

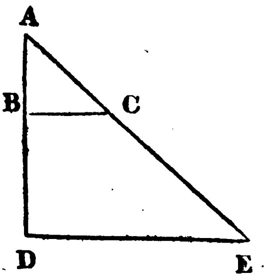


Let AB and AC make any angle at the point A, and in AC take any number of equal parts CE, ED, DL, LA. Join BC, and through the points E, D, L, draw the straight lines EG, DF, LK, parallel to BC. Then AB will be divided into the same number of equal parts as AC, for the parts in AB are proportional to those in AC. Ed.

PROPOSITION XI. PROBLEM.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.



Produce AB, AC to the points D, E; make BD equal to AC; join BC, and through D draw DE parallel to BC.

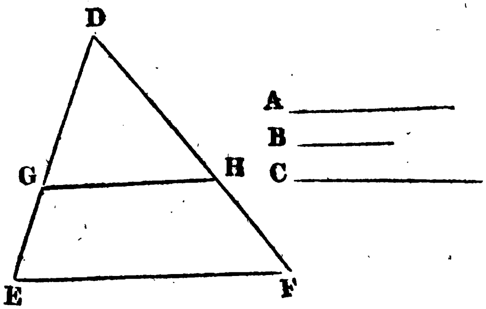
Because BC is parallel to DE, a side of the triangle ADE, AB : BD : : AC : CE (2. 6). But BD = AC; therefore AB : AC : : AC : CE. Wherefore to the two given lines AB, AC a third proportional CE is found. Which was to be done.

PROPOSITION XII. PROBLEM.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any angle EDF, and on these make DG = A, GE = B, and DH = C; join GH, and through the point E draw EF parallel to GH. Because



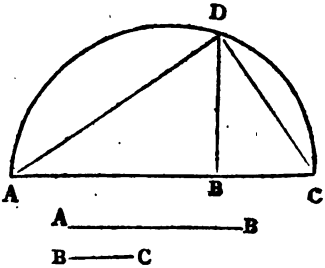
GH is parallel to EF, one of the sides of the triangle DEF, DG : GE : : DH : HF (2. 6). But DG = A, GE = B, and DH = C; therefore A : B : : C : HF. Wherefore to the three given lines A, B, C a fourth proportional HF is found. Which was to be done.

PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line AC, and on AC describe the semicircle ADC; from the point B draw BD perp. to AC, and join AD, DC. Then BD is the mean proportional required.



Because ADC is a right angle (31. 3), and because DB is perp. to AC, DB is a mean proportional between AB, BC, the segments of the base (1 Cor. 8. 61). Therefore between the two given straight lines AB, BC a mean proportional DB is found. Which was to be done.

Cor. Hence it is manifest that if a straight line be drawn from any point in the circumference of a circle perp. to the diameter, it will be a mean proportional between the segments of the diameter. Ed.

PROPOSITION C. THEOREM. Ed.

If a straight line be harmonically divided, the sum of the whole line and one extreme will be to the sum of the other extreme and the middle segment, as the first extreme is to the middle segment.

Let the straight line AB be harmonically divided in two points C and D; AB + BC : AC : : BC : CD.

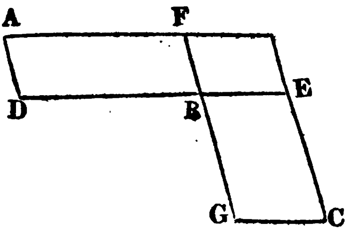
Because AB : BC : : AD : DC (Def. C. 6),
AB : AD : : BC : DC (Propor. 36), therefore
AB+BC : AC : : BC : DC (Propor. 41).

Therefore, if a straight line &c. Q.E.D.

PROPOSITION XIV. THEOREM.

Equal parallelograms, which have one angle of one parallelogram equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and parallelograms which have one angle of one parallelogram equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

1. Let AB, BC be two equal parallelograms, which have the angles at B equal; the sides about the equal angles are reciprocally proportional, that is, DB is to BE as GB to BF.



Let the sides DB, BE be placed in the same line, then the angles DBG, GBE are equal to two right angles (13. 1), therefore the angles DBG, DBF are equal to two right angles, because the angles DBF, GBE are equal; therefore FBG is a straight line. Complete the paral. FE. Because the parals. AB, BC are equal, and FE is another parallelogram,

AB : FE : : BC : FE.
But AB : FE : : DB : BE (1. 6),
and BC : FE : : GB : BF;
therefore DB : BE : : GB : BF (Propor. 34.)

Wherefore the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.

2. Let the sides about the equal angles be reciprocally proportional, DB to BE as GB to BF; the parallelograms AB, BC are equal.

Because DB : BE : : GB : BF,
and DB : BE : : AB : FE,
and GB : BF : : BC : FE,
AB : FE : : BC : FE (Propor. 34);

wherefore the parallelograms AB, BC are equal. Therefore, equal &c. Q. E. D.

Cor. 1. If two triangles, which have one angle of one triangle equal to one angle of the other, be equal to each other, the sides about the equal angles are reciprocally proportional; and conversely. For a triangle is equal to half a paral. of the same base and altitude (41. 1). Ed.

Cor. 2. Equal rectangles have their sides reciprocally proportional; and conversely. Ed.

Cor. 3. Hence if four straight lines be proportional the rectangle contained by the extremes is equal the rectangle contained by the means; and conversely. Ed.

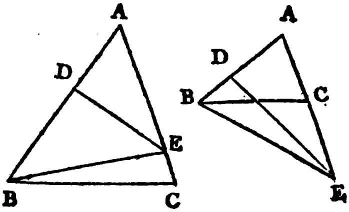
Cor. 4. If three straight lines be proportional the rectangle contained by the extremes is equal to the square of the mean; and conversely. Ed.

Cor. 5. Equal triangles, or equal parallelograms, have their bases and altitudes reciprocally proportional; and conversely. Ed.

PROPOSITION D. THEOREM. Legendre.

Two triangles which have one angle of one triangle equal to one angle of the other are to each other as the rectangles contained by the sides about the equal angles.

Let ABC, ADE, be two triangles which have the angles at A equal; the triangle ABC : triangle ADE : : AB . AC : AD . AE.



Join BE. The triangles ABE, ADE, whose common vertex is E, have the same altitude, and therefore are to each other as their bases (1. 6), that is, ABE : ADE : : AB : AD. In the same manner ABC : ABE : : AC : AE. Therefore ABC : ADE : : AB.AC : AD . AE (Propor. 42. 43). Therefore, two triangles &c. Q. E. D.

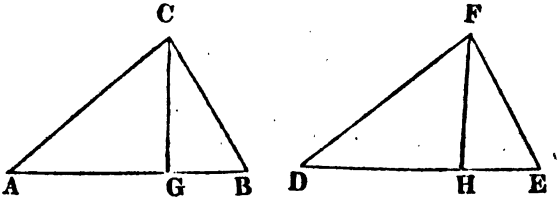
Cor. 1. Two equiangular triangles are to each other as the rectangles contained by the sides about two equal angles.

Cor. 2. Hence, two equiangular parallelograms are to each other as the rectangles contained by the sides about the equal angles.

PROPOSITION XIX. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be two similar triangles, having the angles at A and D equal; and let AC : AB : : DF : DE so that the side AB be homologous to DE (Propor. 22); the triangle ABC : triangle DEF : : $AB^2 : DE^2$.



From the angles C and F draw the perpendiculars CG and FH; then

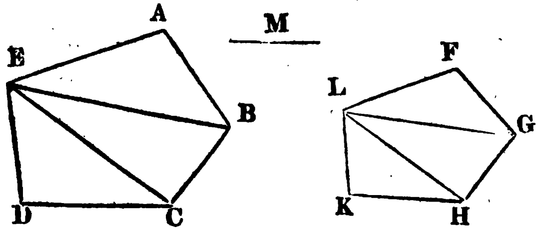
by similar triangles ABC, DEF, AB : DE : : AC : DF (4. 6), and by similar triangles ACG, DFH, CG : FH : : AC : DF; therefore AB . CG : DE . FH : : AC^2 : : DF^2 (Propor. 42). But the rectangle AB . CG is double of the triangle ABC (41. 1), and the rectangle DE . FH is double of the triangle DEF. Consequently the triangle ABC : triangle DEF : : $AC^2 : DF^2$. Therefore, similar triangles &c. Q. E. D. Ed.

Otherwise. The application remaining, the triangle ABC : triangle DEF : : AB . BC : DE . EF (1 Cor. D. 6), and AB : DE : : BC : EF (4. 6). For the ratio of BC : EF in the first proportion substitute the equal ratio of AB : DE, then the triangle ABC : triangle DEF : : AB . AB : DE . DE. Therefore, similar triangles &c. Q. E. D. Ed.

PROPOSITION XX. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another which the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be similar polygons, and let AB be the homologous side to FG; the polygons may be divided into the same number of similar triangles, whereof each has to each the same ratio which the polygons have; and the polygon ABCDE has to FGHKL the duplicate ratio of that which the side AB has to FG.



From any two equal angles E and L draw the lines EB, EC, and LG, LH. Because the polygon ABCDE is similar to FGHKL, the angle BAE is equal to GFL, and BA : AE : : GF : FL (Def. 1. 6); therefore the triangle ABE is equiangular (6. 6), and similar to the triangle FGL; wherefore the angle ABE is equal to FGL.

Because the polygons are similar, the whole angles ABC, FGH are equal; therefore the remaining angles EBC, LGH are equal. Now because the triangles ABE, FGL are similar, EB : BA : : LG : GF; and because the polygons are similar, AB : BC : : FG : GH; therefore
EB : BC : : LG : GH (Propor. 42, 43);
therefore the sides about the equal angles EBC, LGH are proportional; therefore the triangles EBC, LGH are equiangular and similar. For the same reason the triangle ECD is similar to LHK. Therefore the similar polygons ABCDE, FGHKL are divided into the same number of similar triangles.

Again, these triangles have, each to each, the same ratio which the polygons have to each other, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK; and the polygon ABCDE has to FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to FGL, ABE has to FGL the duplicate ratio of that which the side BE has to GL (19. 6). For the same reason the triangle BEC has to GLH the duplicate ratio of that which the side BE has to GL. Therefore the triangle ABE : triangle FGL : : triangle BEC : triangle GLH (Propor. 34).

Again, because the triangle EBC is similar to LGH, EBC has to LGH the duplicate ratio of that which the side EC has to LH. For the same reason the triangle ECD has to LHK the duplicate ratio of that which EC has to LH. Therefore the triangle EBC : triangle LGH : : triangle ECD : LHK (Propor. 34).

But it has been proved that the triangle EBC : triangle LGH : : triangle ABE : triangle FGL. Therefore the triangle ABE : triangle FGL : : triangle EBC : triangle LGH : : triangle ECD : triangle LHK; therefore the triangle ABE : FGL : : polygon ABCDE : FGHKL (Propor. 41).

But the triangle ABE : FGL : : AB^2 : FG^2. Therefore the polygon ABCDE : polygon FGHKL : : AB^2 : FG^2. Wherefore, similar polygons &c. Q. E. D.

Cor. 1. In like manner it may be proved that similar figures of four sides, or of any number of sides, are to one another in the duplicate ratio of their homologous sides; and it has been proved in triangles. Therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Cor. 2. Because all squares are similar figures, the ratio of any two squares to each other is the same with the duplicate ratio of their sides; and hence, also, any two similar rectilineal figures are to each other as the squares of their homologous sides.

Cor. 3. Two similar triangles, or two similar polygons, are to each other as any rectilineal figure described on any side of one is to a similar figure similarly described on the homologous side of the other.

For the two given figures, and the two similar figures thus similarly described, will have to each other the same duplicate ratio of that which their homologous sides have to each other.

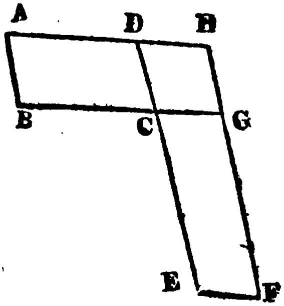
Ed.

PROPOSITION XXIII. THEOREM.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides about the equal angles; "that is, they are to one another as the rectangles contained by the sides about the equal angles."

Let AC, CF be two equiangular parallelograms, having the angle BCD equal to ECG; the ratio of the paral. AC to the

paral. CF is the same with the ratio which is compounded of the ratios of their sides about the equal angles; that is, the paral. AC : paral. CF : : BC . CD : EC . CG.



Let BC, CG be placed in a straight line, then, because the angles GCE, BCD are equal, DC, CE are also in a straight line (15. 1). Complete the parallelogram DG. The paral. AC : paral. CH : : BC : CG (1. 6), and the paral. CH : paral. CF : : DC : CE; therefore the paral. AC : paral. CF : : BC . CD : CG . CE (Propor. 42, 43). Therefore, equiangular &c. Q. E. D. Ed.

Cor. 1. Equiangular triangles have to one another the ratio which is compounded of the ratios of their sides about the equal angles. Ed.

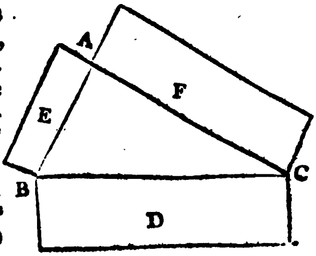
Cor. 2. Two parallelograms, which have one angle of one equal to one angle of the other, are to each other in a ratio compounded of the ratios of the sides of one to the sides of the other, each to each, about their equal angles.

For two parals. having one angle of one paral. equal to one angle of the other are mutually equiangular (E. 1).

PROPOSITION XXXI. THEOREM.

In a right angled triangle if similar rectilineal figures be similarly described on the three sides, the figure on the hypothenuse will be equal to both the figures on the other two sides.

Let ABC be a right angled triangle, and BAC the right angle; the rectilineal figure D described on the hypothenuse BC is equal to the similar figures E, F, similarly described on BA, AC.



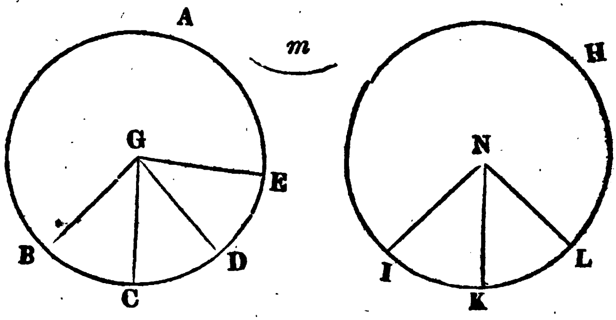
Because similar polygons are as the squares of their homologous sides (20. 6), $E : AB^2 :: F : AC^2 :: D : BC^2$, $\therefore E + F :: AB^2 + AC^2 :: D : BC^2$ (Propor. 41). But $AB^2 + AC^2 = BC^2$ (B. 2), therefore $E + F = D$ (Propor. 32). Therefore, in a right &c. Q. E. D. Lacroix.

PROPOSITION XXXIII. THEOREM.

In equal circles angles either at the centres, or at the circumferences, have the same ratio to one another as the arches on which they stand have to one another.—*See Note.*

Let ABC, HIK be two equal circles, and let BGE, INL be angles at their centres; the arch BE : IL : : angle BGE : INL.

Let the arches BE, IL be commensurable, and let the small arch *m* be their common measure. Divide the arch BE into parts BC, CD, DE, each equal to *m*; and divide the arch IL into parts IK, KL, each equal to *m*. From the centres G and N draw straight lines to the several points of division.



The angles BGC, CGD, &c. of both the circles are equal to one another, because the arches BC, CD, &c. on which they stand are all equal (27. 3). The angle BGE contains BGC as often as the arch BE contains *m*, and the angle INL contains INK, or its equal BGC, as often as the arch IL contains *m*. Consequently the angle BGE : BGC : : arch BE : *m* (Propor. 19), and the angle INL : INK : : arch IL : *m*. These two analogies become, by alternation (Propor. 36), the angle BGE : arch BE : : angle BGC : *m*, and the angle INL : arch IL : : angle BGC : *m*. Hence the angle BGE : arch BC : : angle INL : arch IL (Propor. 34), ∴ angle BGE : angle INL : : arch BE : arch IL.

Next, because an angle at the circumference of a circle is equal to half the angle at the centre on the same arch (20. 3), half BGE : half INL : : BE : IL (Propor. 30). Therefore, in equal circles &c. Q. E. D. ED.

Cor. 1. Hence in the same circle the angles at the centre, or at the circumference, are as the arches on which they stand: consequently arches may be assumed as the measures of the magnitudes of angles. Ed.

Cor. 2. An angle at the centre of a circle is to four right angles as the arch on which it stands is to the whole circumference of the circle.

For any angle BGC is to a right angle as the arch BC on which it stands is to a quadrant. But a right angle is one fourth of all the angles about the centre of a circle (3 Cor. 13. 1), and a quadrant is one fourth of the whole circumference. Therefore the angle BGC is to four right angles as the arch BC is to four quadrants, or the whole circumference of the circle (Propor. 30). Ed.

Cor. 3. In unequal circles the arches which subtend equal angles at the centres are to one another as the circumferences of the circles are to one another.

Let A and B denote two unequal circles, then any angle at the centre of A is to four right angles as the arch on which it stands is to the circumference of A (Cor 2); also an equal angle at the centre of B is to four right angles as the arch on which it stands is to the circumference of B. Therefore the arch of A is to the circumference of A as the similar arch of B is to the circumference of B (Propor. 34). Ed.

Cor. 4. Hence, if the circumferences of any two unequal circles be divided into the same number of equal parts, whatever number of those parts is contained in any arch of one circle subtending a given angle, the same number of parts will be contained in a similar arch of the other circle, subtending an angle equal to the former. Ed.

Scholium. It is in consequence of the ratio which subsists between angles at the centre of the same circle, or of equal circles, and the arches which subtend them, that an arch of a circle is called the measure of its corresponding angle. The circumference of the circle is the only curve whose arches increase or decrease in the ratio of the corresponding angles at the centre.

If the circumference of any circle be supposed to be divided into any number of equal parts, as 360, the number of parts contained in any arch BC, which is intercepted by two radii BG, CG, will be the proper measure of the angle BGC.

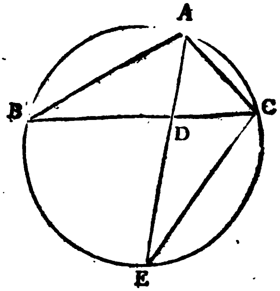
Mathematicians suppose the circumference of every circle to be divided into 360 equal parts, called *degrees*, and each degree into 60 equal parts, called *minutes*, and each minute into 60 equal parts, called *seconds*, &c.

Hence a right angle, being measured by a quadrant contains 90 degrees, the three angles of a triangle, or two right angles, contain 180 degrees, and half a right angle contains 45 degrees. An acute angle is measured by an arch which is less than 90 degrees, and an obtuse angle by an arch which is greater than 90 degrees.

PROPOSITION E. THEOREM. SIMSON.

If an angle of a triangle be bisected by a straight line which cuts the opposite side, the square of that line together with the rectangle contained by the segments of the base are equal to the rectangle contained by the other two sides.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; the square of AD together with the rectangle BD . DC are equal to the rectangle BA . AC.



Suppose the circle ACB to be described about the triangle; produce AD to the circumference in E, and join AC. Then, because the angle BAD is equal to CAE, and the angle ABD to AEC (21. 3), the triangles ABD, AEC are equiangular to each other (2 Cor. 32. 1); therefore BA : AD : : EA : AC (4. 6), consequently BA . AC = AD . AE = ED . DA + DA² (3. 2).* But ED . DA = BD . DC (35. 3); therefore BA . AC = BD . DC + DA². Wherefore, if an angle &c. Q. E. D.

COR. 1. If a circle be described about a triangle, and a chord be drawn to bisect any angle of the triangle, the rectangle contained by the sides about that angle is equal to the rectangle contained by the chord and its segment intercepted between the vertex and the base of the triangle; or to the rectangle contained by the segments of the chord together with the square of the segment between the vertex and the base of the triangle.

For BA . AC = AD . AE = AD . DE + AD², by the demonstration. ED.

COR. 2. If the triangle ABC be isosceles, then AB² = BD² + AD², because the triangles ABD, ACD will be equal in all respects (4. 1), or AB² = AD . AE, or AD . DE + AD². ED.

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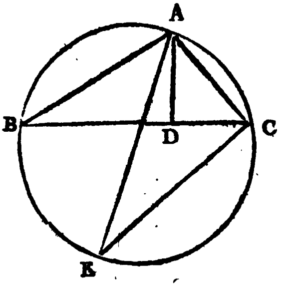
* AD . AE = AD . (AD + DE) = AD² + AD . DE.

PROPOSITION F. THEOREM. SIMSON.

If a circle be described about a triangle, and a straight line be drawn from any angle perpendicular to the opposite side, the rectangle contained by the other two sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle.

Let ABC be a triangle, and AD the perp. drawn from the angle A to the opposite side BC; the rectangle BA . AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.



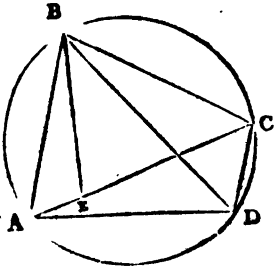
Suppose the circle ACB to be described about the triangle; draw its diameter AE, and join EC. Because the right angle BDA is equal to ECA in a semicircle (31. 3), and the angle ABD to AEC in the same segment (21 3), the triangles ABD, AEC are equiangular; therefore BA : AD : : EA : AC (4. 6), ∴ BA . AC = EA . AD (Propor. 26). Therefore, if a circle &c. Q. E. D.

PROPOSITION G. PROBLEM. SIMSON.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and let the diagonals AC, BD be drawn; the rectangle AC . BD is equal to the two rectangles AB . CD and AD . BC, or AC . BD = AB . CD + AD . BC.

Make the angle ABE equal to DBC; add to each of these the angle EBD, then the angle ABD is equal to EBC. But the angle BDA is equal to BCE (21. 3); therefore the triangle ABD is equiangular to the triangle BCE; wherefore BC : CE :: BD : DA (4. 6), ∴ BC . DA = BD . CE (Propor. 26).



Again, because the angle ABE is equal to DBC, and the angle BAE to BDC, the triangle ABE is equiangular to BCD; therefore BA : AE :: BD : DC, ∴ BA . DC = BD . AE. But BC . DA = BD . CE; wherefore BC . DA + BA . DC = BD . CE + BD . AE = BD . AC (1. 2). Therefore, the rectangle &c. Q. E. D.

PROPOSITION H. THEOREM. Ed.

If two chords of a circle intersect each other, the segments of one chord are reciprocally proportional to the segments of the other.

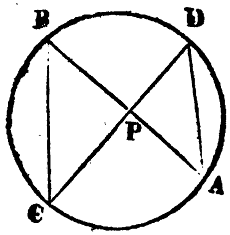
PROPOSITION I. THEOREM. Ed.

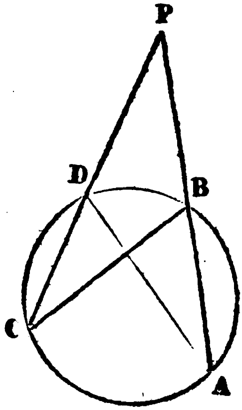
If from any point without a circle two secants be drawn, they are reciprocally proportional to their external segments, or their segments between the point and the convex circumference.

(The above two Propositions are demonstrated together as follows).

Through any point P, either within or without the circle ABC, let two lines AB, CD be drawn to cut or meet the circumference in the points A, B, C, D; then PA : PC :: PD : PB.

Join the points A, D, C, B, where the two chords or secants meet or cut the circle. In the triangles APD, CPB the vertical angles at P are equal, and the angles BAD, BCD are equal, because they stand on the same arch BD (27. 3); therefore, in the first case, the angles ABC, ADC are equal (2 Cor. 32. 1), and in the second case, the angles ADP, CBP





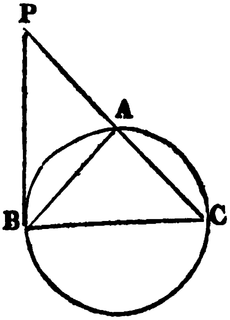
are equal. Hence the triangles ADP, CBP are equiangular, therefore PA : PD : : PC : PB (4. 6), ∴ PA : PC : : PD : PB (Propor. 36). Therefore, if &c. Q. E. D.

PROPOSITION K. THEOREM. Ed.

If from any point without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the secant and its external part.

From any point P without the circle ABC let a tangent PB, and a secant PC be drawn; PC : PB : : PB : PA.

Draw AB, AC; then the triangles APB, CPB, having the angles ABP, PCB equal (32. 3), and the angle P common, are equiangular (2 Cor 32. 1), therefore PC : PB : : PB : PA (4. 6), or AP . PC = PB^2. Therefore, if from any &c. Q. E. D.

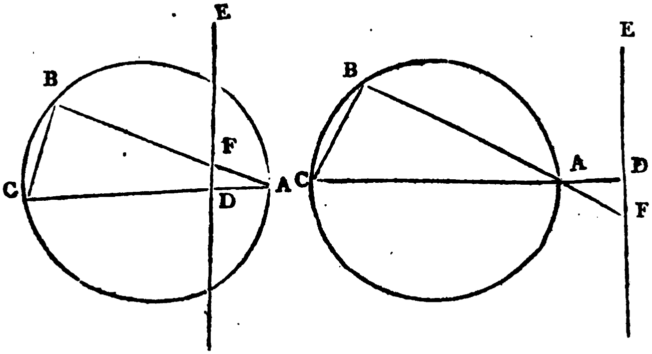


Note. The last three Propositions are the same as Prop. 35, B, 36, Book III.

PROPOSITION L. THEOREM.

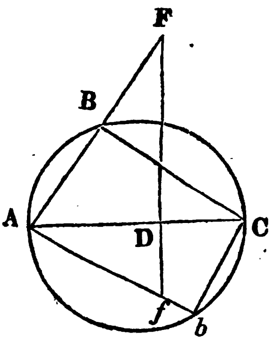
If through one extremity of the diameter of a circle a chord be drawn to meet any perpendicular to the same diameter, either within, or produced without the circle, the chord and the diameter will be reciprocally proportional to their segments intercepted between the same extremity of the diameter and the perpendicular.

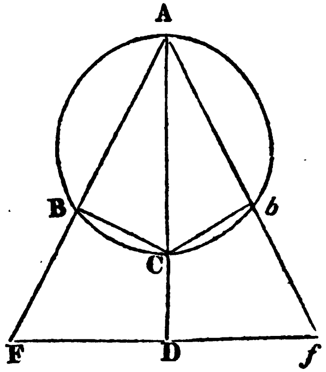
Let ABC be a circle, of which AC is a diameter; let DE be perp. to AC, and let a chord AB meet DE in F; AB : AC : : AD : AF.



Join BC; then the angle ABC is right (31. 3). Now the angle ADF is right; and the angle BAC is equal to DAF, therefore the triangles ABC, ADF are equiangular, therefore BA : AC : : AD : AF. Therefore, if through one extremity &c. Q. E. D.

COR. If through one extremity A of a diameter AC of a circle two chords AB, A*b* be drawn to meet any perpendicular F*f* to the same diameter, either within the circle, or if produced, without it, the two chords will be reciprocally proportional to their segments AF, A*f*. intercepted between the same extremity of the diameter and the perpendicular.



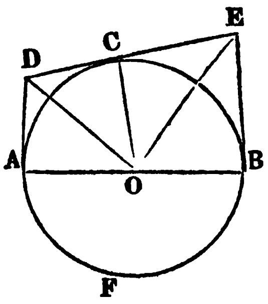


For AB : AC : : AD : AF, and A*b* : AC : : AD : A*f*; therefore AB . AF = AC . AD, and A*b* . A*f* = AC . AD; therefore AB . AF = A*b* . A*f*, ∴ AB : A*b* : : A*f* : AF. Ed.

PROPOSITION M. THEOREM. Ed.

If two tangents be drawn from the extremities of any diameter of a circle, and meet a tangent to any point in the circumference, the radius of the circle will be a mean proportional between the segments of the third tangent intercepted between its point of contact and the other two tangents.

Let the line DE touch the circle ACBF in C, and from the extremities of the diameter AB let the tangents AD, BE be drawn meeting DE in the points D and E; draw OC; then OC is a mean proportional between the segments CD, CE of the tangent DE.



Draw OD, OE. The right angled triangles AOD, COD, having AO = CO, and DO common, have AD = CD, and the angles at D and O equal (2 Cor. B. 2). In the same manner the right angled triangles BOE, COE are equal in all respects. The angles AOC, BOC are together equal to two

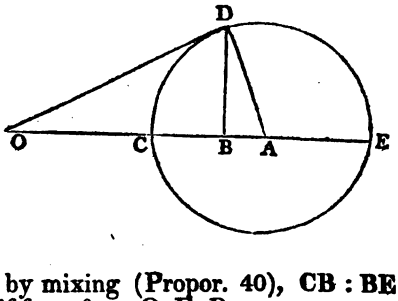
right angles, therefore their halves, or the angles DOC, EOC are together equal to one right angle, that is, DOE is a right angle. The radius OC drawn to the point of contact C of the tangent DE is perp. to DE (18. 3), therefore the triangles DCO, ECO are similar (8. 6), therefore DC : OC : : OC : CE (1 Cor. 8. 6). Therefore, if two tangents &c. Q. E. D.

Cor. It appears from the demon. that if two tangents be drawn from the extremities of any diameter of a circle, and meet another tangent, the angle DOE contained by two straight lines OD, OE drawn from the centre to the points of concourse of the tangents is a right angle.

PROPOSITION N. THEOREM. Ed.

If from any point in a diameter of a circle produced a tangent be drawn, and from the point of contact a perpendicular to the diameter be drawn, the segments of the diameter made by the perpendicular will have the same ratio as the distances of the first point from the extremities of the diameter have to each other.

Let A be the centre of the circle CDE, and from any point O in the diameter CE produced draw a tangent OD; from the point of contact D draw DB perp. to CE; CB : BE : : OC : OE.



Join DA. Because ADO is a right angle (18. 3), and DB is perp. to AO, the triangles ADB, ADO are similar (8. 6), therefore AB : AD : : AD : AO, or AB : AC : : AC : AO, therefore by mixing (Propor. 40), CB : BE : : OC : OE. Therefore, if from &c. Q. E. D.

Cor. In any circle CDE if a tangent DO meet a radius AC produced, and from the point of contact D a perpendicular DB be drawn to the diameter CE, the radius AC will be a mean proportional between the radius produced and the distance of the perpendicular from the centre. For by the demon. AB : AC : : AC : AO.

DEF. 1. The perimeter of any figure is the length of the line, or lines, by which it is bounded.

DEF. 2. The surface of any figure is the space contained within its perimeter.

AXIOM. If two figures have the same straight line for their base, and if one figure be contained within the other, it has the less perimeter, if its bounding line or lines be no where convex toward the base.

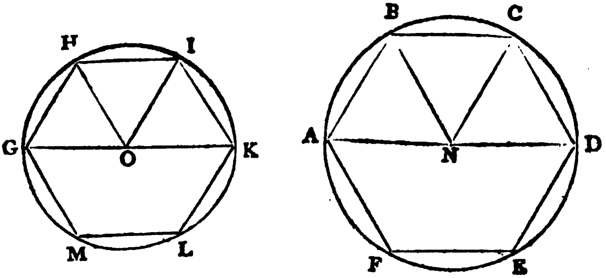
COR. 1. Hence the perimeter of any polygon inscribed in a circle is less than the circumference of the circle.

COR. 2. If from any point two tangents be drawn to a circle, they are together greater than the arch intercepted between them; and hence the perimeter of any polygon described about a circle is greater than the circumference of the circle.

PROPOSITION O. THEOREM.*

Equilateral polygons of the same number of sides, inscribed in circles, are similar, and are to one another as the squares of the diameters of the circles.

Let ABCDEF, GHIKLM be two equilateral polygons of the same number of sides, inscribed in the circles ABD, GHK; they are similar, and are to each other as the squares of the diameters of the circles.



1. Find N, O the centres of the circles; join AN, BN, and GO, HO; produce AN and GO till they meet the circumferences in D and K. Because the straight lines AB, BC, CD, DE, EF, FA are all equal, the arches AB, BC, CD, DE, EF,

* The following propositions are transferred from Playfair's First Supplement to the end of Book VI, to which they properly belong.

FA are also equal (28. 3). For the same reason the arches GH, HI, IK, KL, LM, MG are all equal. Therefore, whatever part the arch AB is of the whole circumference ABD, the same part is the arch GH of the whole circumference GHK. But the angle ANB is the same part of four right angles that the arch AB is of the circumference ABD (2 Cor. 33. 6), and the angle GOH is the same part of four right angles that the arch GH is of the circumference GHK; therefore the angles ANB, GOH being each of them the same part of four right angles are equal to each other. Therefore the isosceles triangles ANB, GOH are equiangular (6. 6), and the angle ABN is equal to GHO.

In the same manner, by joining NC, OI, it may be proved that the angles NBC, OHI are equal to each other, and to the angle ABN. Therefore the whole angle ABC is equal to the whole GHI. The same may be proved of the angles BCD, HIK, and also of the rest. Therefore the polygons ABCDEF, GHIKLM are equiangular to each other; and since they are equilateral, the sides about the equal angles are proportionals; therefore the polygon ABCDEF is similar to the polygon GHIKLM (1 Def. 6).

2. Again, because the polygons ABCDEF, GHIKLM have been proved to be similar, the polygon ABCDEF is to GHIKLM as the square of AB to the square of GH (20. 6). But because the triangles ANB, GOH are equiangular, the square of AB is to the square of GH as the square of AN to the square of GO (4. 6), or as the square of AD to the square of GK (Propor. 30). Therefore also the polygon ABCDEF is to GHIKLM as the square of AD to the squares of GK; and they have also been shown to be similar. Therefore, equilateral polygons &c. Q. E. D.

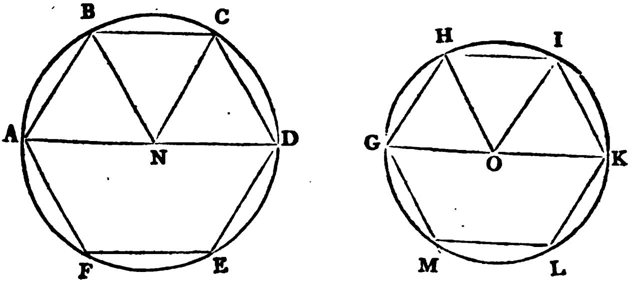
Cor. Every equilateral polygon inscribed in a circle is also equiangular. For the isosceles triangles, which have their common vertex in the centre, are all equal and similar; therefore the angles at their bases are all equal, therefore the angles of the polygons are also equal.

PROPOSITION P. THEOREM.*

The homologous sides, and also the perimeters of similar polygons inscribed in circles, are to one another as the diameters of the circles.

* The following Propositions are taken chiefly from Hutton's Mathematics.

Let ACE, GIL be two similar polygons inscribed in the circles ABD, GHK; the side AB is to GH as the diameter AD



to GK, and the perimeter AB + BC + CD &c. is to the perimeter GH + HI + IK &c. as the diameter AD to GK.

It may be proved as in the last Prop. that the triangles ANB, GOH are equiangular, therefore AB : GH :: AN : GO :: AD : GK.

Again, if straight lines be drawn from the centres N, O of the circles to all the angular points of the polygons, it is evident that they will divide the polygons into the same number of equal and similar triangles, whose homologous sides will be to each other as the radii or diameters of the circles. Hence (Propor. 41), the sum of the sides, or the perimeter of the polygon ACE, is to the perimeter of the polygon GIL, as the diameter AD to the diameter GK. Therefore, the homologous sides &c. Q. E. D.

PROPOSITION Q. THEOREM.

The circumferences of circles are to one another as their diameters.

Let D and d denote the diameters of two circles, and C and c the circumferences; then $D : d :: C : c$, or $D : C :: d : c$.

Let the number of sides of a polygon be indefinitely great, and consequently the length of each side indefinitely small; then will the perimeter of the polygon approach indefinitely near to the circumference of the circle described about it, and therefore may be conceived to coincide with it. But the perimeters of similar polygons of any number of sides inscribed in circles are to one another as the diameters of the circles. Therefore the circumferences of circles are to one another as their diameters. Q. E. D.

COR. Similar arches of unequal circles are to one another as the diameters or radii of the circles. For D : d :: $\frac{1}{2}$D : $\frac{1}{2}d$ :: any part of C : same part of c (Propor. 30).

PROPOSITION R. THEOREM.

The surfaces of circles are to one another as the squares of their diameters.

Let A and a denote the surfaces of two circles, D and d their diameters; then $A : a :: D^2 : d^2$

Let the number of sides of a polygon be indefinitely great, and consequently the length of each side indefinitely small; then the perimeter of the polygon will coincide with the circumference of the circles described about it; therefore their surfaces will become equal. But similar polygons of any number of sides inscribed in circles are to one another as the squares of the diameters of the circles. Hence the surfaces of circles are to one another as the squares of their diameters. Q. E. D.

COR. The surfaces of circles are to one another as the squares of their circumferences. For the circumferences are as the diameters (Q).

PROPOSITION S. THEOREM.

The surface of a circle is equal to a rectangle contained by the radius and a straight line equal to half the circumference.—*See figure, Prop. P.*

Let ABDE be a circle, of which the centre is N, and the diameter AD; the surface of ABDE is equal to a rectangle contained by AN and a line equal to half the circumference ABCD.

Let a regular polygon be inscribed in the circle, and suppose radii to be drawn from the centre to all the angular points, dividing the polygon into as many equal triangles as it has sides. Let ANB be one of the triangles, whose altitude is the perpendicular drawn from the centre on the base AB. The triangle ABN is equal to half the rectangle under the same base and altitude (41. 1), or the rectangle under its altitude and half its base AB; therefore the sum of all the equal triangles which compose the polygon ABC &c. is equal to the rectangle under the common altitude and half the sum of the sides of the polygon.

Now suppose the number of sides of the polygon to be increased indefinitely, and consequently the length of each side to be decreased indefinitely; then will its perimeter coincide with the circumference of the circle, and the perpendicular from the centre on one of its sides will become equal to the radius. Hence the surface of the polygon will be equal to that of the circle. Consequently the surface of a circle is equal to a rectangle under the radius and a straight line equal to half the circumference. Q. E. D.

Cor. 1. A circle is equal to a triangle whose altitude is equal to the radius and base equal to the circumference. This is evident from the Prop. and 41. 1.

Cor. 2. Of all plane figures having equal perimeters the circle is the greatest. For a circle is equal to a triangle whose base is equal to the circumference and altitude equal to the radius (Cor. 1); and, by the demonstration, a polygon is equal to a triangle whose base is equal to the perimeter and altitude equal to the altitude of the polygon. But, by the hypothesis, the perimeter of the polygon is equal to the circumference of the circle; and its altitude is always less than the radius of the circle, which, in this case, is partly within and partly without the polygon. Consequently the surface of the polygon is always less than the surface of the circle.

Cor. 3. Every regular polygon is equal to a rectangle of which the base is equal to half the perimeter of the polygon, and altitude equal to the perpendicular drawn from the centre on one of the sides.

THE PRINCIPAL THEOREMS IN BOOK VI.

Triangles, and also parallelograms, of the same altitude are to one another as their bases.

If a straight line be drawn parallel to one of the sides of a triangle it will cut the other two sides proportionally.

If any angle of a triangle be bisected by a straight line which cuts the base or opposite side, the segments of the base will have the same ratio which the other two sides have to each other.

The sides about the equal angles of equiangular triangles are proportional.

If the sides of two triangles about each of their angles be proportional, the triangles will be equiangular.

If two triangles have one angle of one triangle equal to one angle of the other, and the sides about the equal angles proportional, the triangles will be equiangular.

Equal parallelograms, and also equal triangles, which have one angle of one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and parallelograms, and also triangles, which have one angle of one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

In a right angled triangle if a perpendicular be drawn from the right angle to the hypothenuse, it will divide the triangle into two triangles which are similar to the whole triangle, and also to each other.

In a right angled triangle if a perpendicular be drawn from the right angle to the hypothenuse, it will

be a mean proportional between the segments of the hypothenuse; and each of the sides about the right angle will be a mean proportional between the hypothenuse and the segment adjacent to that side.

If a perpendicular be drawn from any point in the circumference of a sircle to the diameter, it will be a mean proportional between the segments of the diameter.

Similar triangles, and all similar figures, are to one another as the squares of their corresponding sides.

Equiangular parallelograms, and also equiangular triangles, are to one another as the the rectangles under the sides about the equal angles.

If three straight lines be proportional, the first is to the third as any rectilineal figure described on the first is to a similar figure similarly described on the second.

In a right angled triangle if similar rectilineal figures be similarly described on the three sides, the figure on the hypothenuse will be equal to both the figures on the other two sides.

In equal circles angles either at the centres or at the circumferences have the same ratio to one another as the arches on which they stand have to one another.

An angle at the centre of a circle is to four right angles as the arch on which it stands is to the circumference of the circle.

In unequal circles arches which subtend equal angles at the centres are to one another as the circumferences of the circles.

The rectangle under the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles under its two opposite sides.

The homologous sides, and also the perimeters of similar polygons inscribed in circles, are to one another as the diameters of the circles.

Equilateral polygons of the same number of sides,

inscribed in circles, are similar, and are to one another as the squares of the diameters of the circles.

The diameter of a circle is to the circumference as the square of the radius is to the surface of the circle.

The surface of any circle is equal to the rectangle contained by the radius and a straight line equal to half the circumference; or, it is equal to a triangle whose base is equal to the circumference and altitude equal to the radius of the circle.

Of all plane figures having equal perimeters the circle is the most capacious.

The circumferences of circles are to one another as the diameters; and the surfaces of circles are to one another as the squares of the diameters, or circumferences.

END OF BOOK VI.

ELEMENTS OF GEOMETRY.

BOOK XI.

OF THE INTERSECTION OF PLANES.

DEFINITIONS.—*See Notes.*

1. A straight line is perpendicular to a plane, when it is perpendicular to every straight line meeting it in that plane.

2. A plane is perpendicular to a plane, when all the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

3. The inclination of a straight line to a plane is the acute angle contained by that line and another straight line drawn from the point in which the first line meets the plane to the point in which a perpendicular drawn from any point in the first line to the plane meets the plane.

4. The inclination of two planes is the angle contained by two straight lines drawn from any point in the line of their common section perpendicular to that line, one line in one plane, and the other line in the other plane.

5. Two planes are said to have the same, or a like inclination to each other, which two other planes have, when the angles of inclination above defined are equal to each other.

6. A straight line is said to be parallel to a plane, when it has no inclination to the plane, or is equidistant from it.

7. Planes are said to be parallel to one another which are equidistant, or do not meet, though produced ever so far.

8. A solid angle is an angle made by the meeting of more than two plane angles in one point, which are not in the same plane.

PROPOSITION I. THEOREM.

One part of a straight line cannot be in a plane, and another part above it.

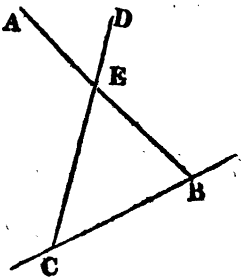
For if one part of a line were in a plane and another part above it, all its parts would not lie in the same direction, and therefore it could not be a straight line (3 Def. 1). Therefore, one part &c. Q. E. D. Ed.

PROPOSITION II. THEOREM.

Any three straight lines which meet one another, but not in the same point, are in one plane.

Let the three straight lines AB, CD, CB meet one another in the points B, C, E; they are in one plane.

Let any plane pass through the line EB, and let the plane revolve round EB as an axis until it pass through the point C; then, because the points E, C are in this plane, the line EC is in it (Def. 8. 1). For the same reason BC is in the same plane; and, by the hypothesis, EB is in it. Therefore the three lines EC, CB, BE are in one plane. But the whole lines DC, AB, BC, produced, are in the same plane with the parts of them EC, EB, BC (1. 11). Therefore AB, CD, CB are all in one plane. Wherefore, any three &c. Q. E. D.

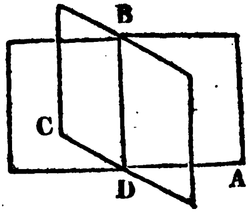


Cor. It is manifest that any two straight lines which cut each other are in one plane; and that any three points whatever are in one plane.

PROPOSITION III. THEOREM.

If two planes cut each other, their common section is a straight line.

Let two planes AB, BC cut each other, and let B, D be two points in the line of their common section. Join BD. Because the points B, D are in the plane AB, the line BD is in the plane AB (Def. 8. 1); and because the points B, D are in the plane BC, the line BD is in the plane BC; therefore the line BD is common to the planes AB and BC, or it is the common section of these planes. Therefore, if two planes &c. Q. E. D.

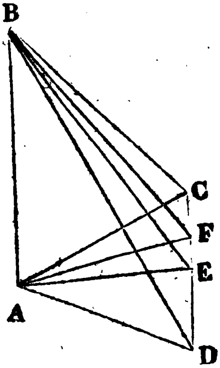


PROPOSITION IV. THEOREM.

If a straight line be perpendicular to each of two straight lines in their point of intersection, it will also be perpendicular to the plane in which those lines are situate.

Let the line AB be perp. to each of the lines AC, AD in their point of intersection A; AB is also perp. to the plane passing through those lines.

Through A draw the line AE bisecting the angle CAD; and also draw any line AF. Take AC = AD; draw CD meeting the lines AE, AF in the points E and F; join BC, BD, BE, BF.



Because the side AC is equal to AD, and AE bisects the angle CAD, the side CE = DE, and AEC, AED are right angles (Cor. 10. 1). Consequently $AD^2 = AE^2 + DE^2$ (B. 2). In the right angled triangles ABC, ABD the side AC = AD, and AB is common, therefore the side BC = BD (2 Cor. B. 2). Hence the triangles BCE, BDE are equal in all respects (8. 1), therefore the angles BEC, BED are equal, and therefore are right angles. Consequently $BD^2 = BE^2 + DE^2$. From these equals take the former equals $AD^2 = AE^2 + DE^2$, then the remainders are equal, that is, $BD^2 - AD^2 = BE^2 - AE^2$, or $AB^2 = BE^2 - AE^2$; wherefore BAE is a right angle (1 Cor. B. 2), and AB is perp. to AE. Now AEF, BEF are right an-

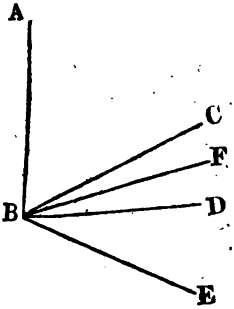
gled triangles, and therefore $AF^2 = AE^2 + EF^2$, and $BF^2 = BE^2 + EF^2$. Hence, by taking the former equals from the latter, $BF^2 - AF^2 = BE^2 - AE^2$, or $BF^2 - AF^2 = AB^2$; therefore BAF is a right angle, and AB is perp. to AF. But AF is any line drawn through the point A; therefore AB is perp. to the plane which passes through the lines AC, AD (Def. 1). Therefore, if a straight &c. Q. E. D. NULTY.

PROPOSITION V. THEOREM.

If three straight lines meet in one point, and a straight line be perpendicular to each of them in that point, those three lines are in one and the same plane.

Let the line AB be perp. to each of the three lines BC, BD, BE, at the point B where they meet; BC, BD, BE are in one and the same plane.

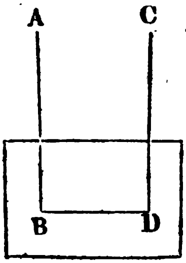
If not; let, if it be possible, BD and BE be in one plane, and BC above it; and let a plane pass through AB, BC; then the common section of this plane with the plane in which BD and BE are situate will be a straight line (3. 11). Let this line be BF; therefore AB, BC, BF are in one plane, which passes through AB, BC.



Because AB is perp. to each of the lines BD, BE, it is also perp. to the plane passing through them (4. 11), and therefore is perp. to every line meeting it in that plane. But BF meets AB in that plane; therefore ABF is a right angle. But, by the hypothesis, ABC is a right angle; therefore the angle ABF is equal to ABC; that is, because these angles are both in the same plane, a part is equal to the whole, which is impossible. Therefore BC is not above the plane in which BD and BE are situate; wherefore BC, BD, BE are in one and the same plane. Therefore, if three &c. Q. E. D.

PROPOSITION VI. THEOREM.

If two straight lines be perpendicular to the same plane they will be parallel to each other.



Let the lines AB, CD be perp. to the same plane BDE; AB is parallel to CD.

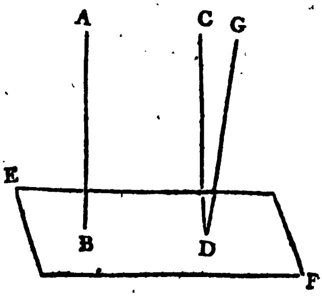
Let the lines AB, CD meet the plane in the points B, D; and let B, D be joined. Because AB, CD are perp. to the same plane, they are also perp. to the line BD in that plane (Def. 1. 11); therefore they are parallel to each other (Cor. 28. 1). Q. E. D.

Cor. If two straight lines be parallel, and if one of them be perp. to any plane, the other also will be perp. to the same plane.

PROPOSITION VII. THEOREM.

If two straight lines be parallel, and if one of them be perpendicular to a plane, the other also is perpendicular to the same plane.

Let AB, CD be two parallel lines, and let one of them AB be perp. to a plane EF; the other CD is perp. to the same plane.



For, if CD be not perp. to the plane EF to which AB is perp. let DG be perp. to it; then DG is parallel to AB (6. 11). Therefore DG and DC are both parallel to AB, and are drawn through the same point D, which is impossible (Ax. 10, p. 40). Therefore, if two &c. Q. E. D.

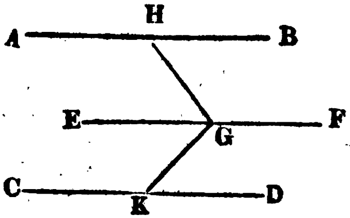
PROPOSITION VIII. THEOREM.

Two straight lines which are parallel to the same straight line, and are not both in the same plane with it, are parallel to each other.

Let AB, CD be each of them parallel to EF, and not in the same plane with it; AB is parallel to CD.

In EF take any point G, from which draw, in the plane passing through EF, AB, the line GH perp. to EF; and in the

plane passing through EF, CD, draw GK perp. to EF. Because EF is perp. to GH and GK, it is perp. to the plane HGK passing through them (4. 2. Sup.). But EF is parallel to AB; therefore AB is perp. to the plane HGK (7. 2. Sup.). For the same reason CD is at right angles to the plane HGK. Therefore AB, CD are perp. to the plane HGK; therefore AB is parallel to CD (6. 2. Sup.). Wherefore, two straight lines &c. Q. E. D.

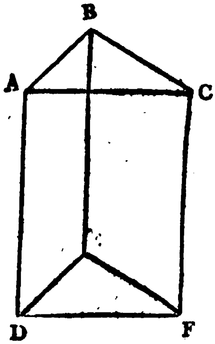


PROPOSITION IX. THEOREM.

If two straight lines which meet each other be parallel to two other straight lines which meet each other toward the same parts, though not in the same plane with the first two, the first two lines and the other two will contain equal angles.

Let the two straight lines AB, BC, which meet each other, be parallel to the two straight lines DE, EF, which meet each other toward the same parts as AB, BC, and are not in the same plane with AB, BC; the angle ABC is equal to the angle DEF.

Take BA = ED, and BC = EF; and draw AD, CF, BE, AC, DF. Because BA is equal and parallel to ED, AD is equal and parallel to BE (33. 1); and because BC is equal and parallel to EF, CF is equal and parallel to BE. Therefore AD and CF are equal and parallel to BE; therefore AD is equal and parallel to CF (8. 2 Sup.); therefore AC is equal and parallel to DF. Now because AB, BC are equal to DE, EF, and the base AC to DF, the angle ABC is equal to DEF (8. 1). Therefore, if two straight &c. Q. E. D.

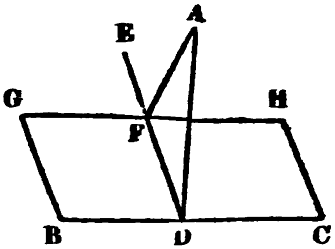


PROPOSITION X. PROBLEM.

To draw a straight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH: it is required to draw from the point A a straight line perp. to the plane BH.

In the plane draw any straight line BC, and from the point A draw AD perp. to BC: then if AD be also perp. to the plane BH the thing required is done. But if it be not, from the point D draw, in the plane BH, the straight line DE perp. to BC; and from the point A draw AF perp. to DE; and through F draw GH parallel to BC.



Because BC is perp. to ED and DA, it is perp. to the plane passing through ED, DA (4. 11). But GH is parallel to BC; wherefore GH is perp. to the plane passing through ED, DA (7. 11), and is therefore perp. to every straight line meeting it in that plane (1 Def. 11). But AF, which is in the plane passing through ED, DA, meets GH; therefore GH is perp. to AF. But AF is perp. to DE. Hence AF is perp. to each of the lines GH, DE; therefore AF is perp. to the plane BH, which passes through ED, GH. Therefore from the given point A above the plane BH the straight line AF is drawn perp. to that plane. Which was to be done.

Cor. If it be required to erect a perpendicular to a plane from a point C in the plane, take a point A above the plane, and draw AF perp. to the plane; then from C draw a line parallel to AF, and it will be the perpendicular required (7. 11).

PROPOSITION XI. THEOREM.

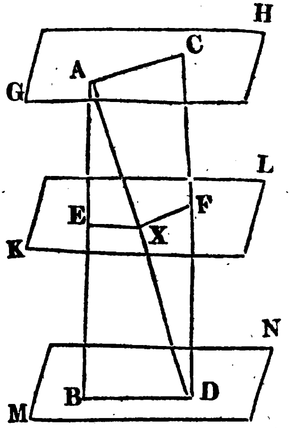
If two parallel planes be cut by a third plane their sections with it are parallel lines.

For if the lines be not parallel, they must meet if produced; and if they meet, the planes in which they are situate must meet, which is impossible (14 Def. 1). Therefore, if two parallel &c. Q. E. D.

PROPOSITION XII. THEOREM.

If two straight lines be cut by parallel planes they will be cut in the same ratio.

Let the lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B, and C, F, D; AE : EB : : CF : FD.



Draw AC, BD, AD; let AD meet the plane KL in the point X; draw EX, XF. Because the parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel (11. 11); and because the parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel.

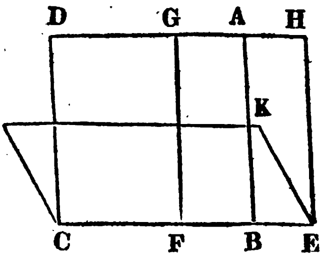
Because the line EX is parallel to BD, a side of the triangle ABD, AE : EB : : AX : XD (2. 6); and because XF is parallel to AC, a side of the triangle ADC, AX : XD : : CF : FD. Therefore AE : EB : : CF : FD (Propor. 34). Wherefore if two &c. Q. E. D.

PROPOSITION XIII. THEOREM.

If a straight line be perpendicular to a plane, every plane which passes through the line is also perpendicular to that plane.

Let the line AB be perp. to a plane CK; every plane which passes through AB is perp. to the plane CK.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE perp. to CE.



Because AB is perp. to the plane CK, it is also perp. to every line meeting it in that plane (1 Def. 11); consequently it is perp. to CE; wherefore ABF is a right angle. But GFB is a right angle; therefore AB is parallel to FG (Cor. 28. 1). But AB is perp. to the plane CK; therefore FG is also perp. to CK (7. 11).

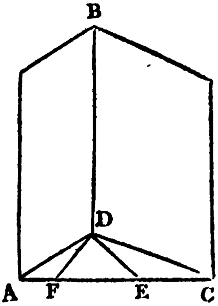
Now any straight line FG in the plane DE, which is perp. to CE the common section of the planes CK, DE, has been proved to be perp. to the other plane CK; therefore the plane DE is perp. to the plane CK (2 Def. 11). In like manner it may be proved that all the planes which pass through AB are perp. to the plane CK. Therefore, if a straight &c. Q. E. D.

PROPOSITION XIV. THEOREM.

If two planes which cut each other be perpendicular to a third plane, their common section will be perpendicular to the third plane.

Let the two planes AB, BC be perp. to a third plane ADC, and let BD be the common section of AB, BC; BD is perp. to the plane ADC.

From D in the plane ADC draw DE perp. to AD, and DF to DC. Because DE is perp. to AD, the common section of the planes AB and ADC, and because the plane AB is perp. to ADC, DE is perp. to the plane AB (2 Def. 11), and therefore also to the line BD in that plane (1 Def. 11). For the same reason DF is perp. to DB. Since BD is perp. to both the lines DE and DF, it is perp. to the plane in which DE and DF are situate, that is, to the plane ADC (4. 11). Wherefore, if two planes &c. Q. E. D.



THE PRINCIPAL THEOREMS IN BOOK XI.

Any two straight lines which cut each other are in one plane; and any three straight lines which meet one another in different points are in one plane.

If two planes cut each other, their common section is a straight line.

If a straight line be perpendicular to each of two straight lines in their point of intersection, it will also be perpendicular to the plane in which those lines are situate.

If three straight lines meet in one point, and a straight line be perpendicular to each of them in that point, those three lines are in one and the same plane.

If two straight lines be perpendicular to the same plane they will be parallel to each other.

If two straight lines be parallel, and if one of them be perpendicular to a plane, the other also is perpendicular to the same plane.

Two straight lines which are parallel to the same straight line, and are not both in the same plane with it, are parallel to each other.

If two straight lines which meet each other be parallel to two other straight lines which meet each other toward the same parts, though not in the same plane with the first two, the first two lines and the other two will contain equal angles.

If two parallel planes be cut by a third plane, their common sections with it are parallel.

If two straight lines be cut by parallel planes they will be cut in the same ratio.

If a straight line be perpendicular to a plane, every plane which passes through the line is also perpendicular to that plane.

END OF BOOK XI.

ELEMENTS OF GEOMETRY.

BOOK XII.

OF THE PROPERTIES OF SOLIDS.

DEFINITIONS.

1. A solid is that magnitude which has length, breadth and thickness.

2. Similar solid figures are such as are contained by the same number of similar planes similarly situate, and having like inclinations to one another.

3. A pyramid is a solid figure contained by more than two triangular planes that are constituted between one plane base and a point above it in which the planes meet.

4. A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to each other; and the rest are parallelograms.

Note.—Prisms and pyramids take particular names according to the figure of their bases. Thus, if the base be a triangle, it is called a triangular prism or pyramid; if a square, it is called a square prism or pyramid. Ed.

5. A parallelopiped is a prism, or solid figure, contained by six quadrilateral figures, whereof every opposite two are parallel.

6. A cube is a solid figure contained by six equal squares.

7. A sphere is a solid figure described by the revolution of a semicircle about a diameter, which remains unmoved: or, a sphere is a solid figure bounded by one curve surface, which is every where equally distant from a certain point within it called the centre.

8. The axis of a sphere is the fixed straight line about which the semicircle revolves.

9. The centre of a sphere is the same with that of a semicircle by which it is described.

10. The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

11. A cone is a solid figure described by the revolution of a straight line round the circumference of a circle, one end of which line is fixed at a point above the plane of the circle.

12. The axis of a cone is the straight line joining the vertex, or fixed point, and the centre of the circle about which the cone is described.

13. The base of a cone is the circle about which the describing line revolves.

14. A cylinder is a solid figure described by the revolution of a rectangle about one of its sides, which remains fixed.

15. The axis of a cylinder is the fixed straight line about which the rectangle revolves.

16. The bases of a cylinder are the circles described by the two revolving opposite sides of the rectangle.

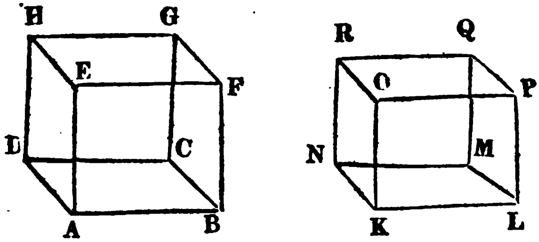
17. Similar cones and cylinders are those which have their axis and the diameters of their bases proportionals.

PROPOSITION I. THEOREM.

If two solids be contained by the same number of equal and similar planes, similarly situated, and if the inclination of any two contiguous planes in one solid be the same with the inclination of the two equal and similarly situated planes in the other, the solids are equal and similar.

Let AG, KQ be two solids contained by the same number of equal and similar planes, similarly situated, so that the plane AC is similar and equal to KM, the plane AF to KP, BG to LQ, GD to QN, DE to NO, and FH to PR; and let the inclination of the plane AF to AC be the same with the inclination of the plane KP to KM, and so of the rest; the solid KQ is equal and similar to the solid AG.

Let the solid KQ be applied to AG, so that the bases KM and AC, which are equal and similar, may coincide (8 Ax. 1), the point N coinciding with the point D, K with A, L with B, and M with C. Because the plane KM coincides with AC,



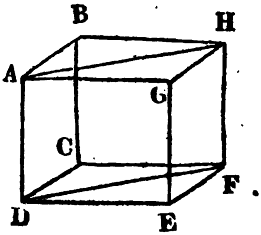
and, by hypothesis, the inclination of KR to KM is the same with the inclination of AH to AC, the plane KR will be on the plane AH, and will coincide with it, because they are similar and equal, and because their equal sides KN and AD coincide. And in the same manner it may be shown that the other planes of the solid KQ coincide with the other planes of the solid AG, each with each. Wherefore the solids KQ and AG wholly coincide, and are equal and similar to each other. Therefore, if two solids &c. Q. E. D.

PROPOSITION II. THEOREM.

If a solid be contained by six planes, of which two and two are parallel, the opposite planes are similar and equal parallelograms.

Let the solid CDGH be contained by the parallel planes AC and GF, BG and CE, FB and AE; its opposite planes are similar and equal parallelograms.

Because the two parallel planes BG, CE are cut by the plane AC, their common sections AB, CD are parallel (11. 11); and because the two parallel planes BF, AE are cut by the plane AC, their common sections AD, BC are parallel. Hence AC is a parallelogram. In like manner it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram.



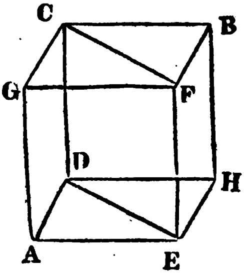
Draw AH, DF. Because AB is parallel to DC, and BH to CF, the angle

ABH is equal to DCF (9. 11). Because AB, BH are equal to DC, CF, and the angle ABH is equal to DCF, the base AH is equal to DF, and the triangle ABH to DCF. For the same reason the triangle AGH is equal to DEF. Therefore the parallelogram BG is equal and similar to CE. In the same manner it may be proved that the parallelogram AC is equal and similar to GF, and the parallelogram AE to BF. Therefore, if a solid &c. Q. E. D.

PROPOSITION III. THEOREM.

If a solid parallelopiped be cut by a diagonal plane, that is, by a plane passing through the diagonals of two of the opposite planes, it will be divided into two equal prisms.

Let AB be a solid parallelopiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, namely, those which are drawn between the equal angles in each. Because CD, FE are parallel to GA, though not in the same plane with it, they are parallel to each other (8. 11); wherefore the diagonals CF, DE are in the same plane as the parallels, and are also parallel (11. 11). The plane CDEF will divide the solid AB into two equal parts.

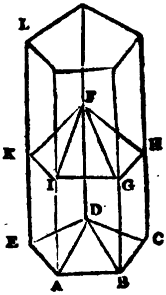


The triangle CGF is equal to CBF (34. 1), and the triangle DAE to DHE; and the parallelogram CA is equal and similar to the opposite one BE (2. 12), and the parallelogram GE to CH; therefore the corresponding planes which contain the prisms CAE, CBE, are equal and similar, each to each. They are also equally inclined to each other, because the planes AC, EB are parallel, and also AF, BD, and they are cut by the plane CE. Therefore the prism CAE is equal to the prism CBE (1. 12), and the solid AB is divided into two equal prisms by the plane CDEF. Therefore, if a solid &c. Q. E. D.

PROPOSITION IV. THEOREM.

If a prism be cut by a plane parallel to the base the section will be equal and like to the base

Let LB be a prism, of which the base is ABCDE, and a section parallel to the base is IGHFK; the section is equal to and like the base.



Draw AD, DB, and IF, FG. Because the planes IGHFK and ABCDE are parallel (by hyp.), and the plane AF cuts them, the sections IF and AD are parallel (11. 11). For the same reason the sections GF and BD are parallel. Because DF and AI are parallel to EK (4 Def. 11), DF is parallel to AI (8. 11). In the same manner DF is parallel to BG. Consequently AF and BF are parallelograms, therefore the side AD = IF (34. 1), and the side BD = GF.

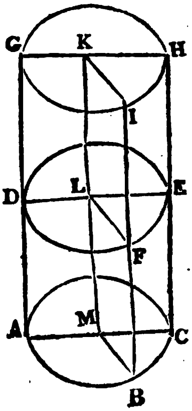
Because the two lines AD, BD meeting each other in D are parallel to the two lines IF, GF meeting each other in F, the angle ADB is equal to IFG (9. 11). Hence the triangles ADB, IFG are equal in all respects (4. 1.).

In the same manner it may be proved that each triangle in the section IGHFK is equal to its corresponding triangle in the base ABCDE. Therefore the sum of all the triangles in the section is equal to the sum of all the triangles in the base, or the section is equal to the base. Also the base and section are alike, because the triangles of which they consist are equal in all respects. Therefore, if a prism &c. Q. E. D.

PROPOSITION V. THEOREM.

If a cylinder be cut by a plane parallel to the base, the section will be a circle, and equal to the base

Let AH be a cylinder, of which the base is ABC, and a section parallel to the base is DFE; the section is a circle equal to the circular base ABC.

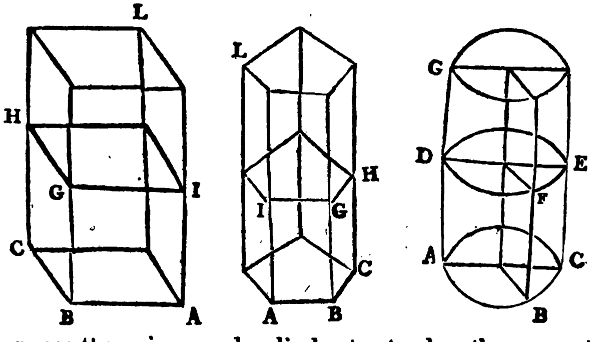


Let the planes MKIB, MKHC pass through MK, the axis of the cylinder, and meet the section DFE in the points L, F, E. Because the plane MKIB cuts the parallel planes ABC, DFE, the sections MB, LF are parallel (11. 11); and because the plane MKHC cuts the same planes, the sections LE, MC are parallel. Now CH and BI are parallel to MK (14 Def. 12), therefore CE and BF are parallel to ML, therefore MF and ME are parallelograms, therefore LF = MB, and LE = MC. But MB = MC, therefore LF = LE. In the same manner it may be proved that all straight lines drawn from the point L to the circumference of the section DFE are equal to one another. Therefore DFE is a circle, and it is equal to the circular base ABC (1 Def. 3). Therefore, if a cylinder &c. Q. E. D.

PROPOSITION VI. THEOREM.

Prisms and cylinders of equal bases and altitudes are all equal to one another.

Let LB, LB be two prisms, and GB a cylinder, having equal bases ABC, and equal altitudes; the two prisms are equal to each other and to the cylinder.



Suppose the prisms and cylinder to stand on the same plane, and be cut by planes parallel to their bases, making the corresponding sections IGH, DFE. The section IGH is equal to the base ABC of the prisms (4. 12), and the section DFE is equal to the base ABC of the cylinder (5. 12).

But these bases are equal (by hyp.), therefore the sections IGH, DFE are equal. For the same reason every section in each of the prisms is equal to the corresponding section of the cylinder.

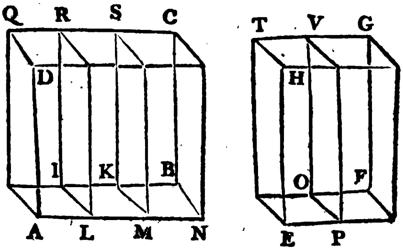
Now if we consider these parallel sections as plates (or laminæ) extremely thin, and their number as indefinitely great, we may conceive the three bodies to be composed of an indefinite number of such elementary plates, of which the aggregate is equivalent to the magnitude or volume of each of the bodies. And since the extent and number of the parallel sections or plates are equal in the several bodies the magnitudes of the bodies must be equal. Therefore, prisms &c. Q. E. D. *See Note.*

Cor. Rectangular parallelopipeds of equal bases and altitudes are equal to one another. For they are prisms (5 Def. 12).

PROPOSITION VII. THEOREM.

Rectangular parallelopipeds of equal altitudes are to one another as their bases.

Let AC, EG be two rectangular parallelopipeds of equal altitudes AD, EH; AC : EG : : base AB : base EF.



Let the bases AB and EF be to each other as any numbers m (3) and p (2), that is, let AB : EF : : m : n. Let AB be divided into m equal rectangles, AI, LK, MB; and EF into n equal rectangles, EO, PF, each equal to the former. Construct the parallelopipeds AR, LS, &c. Because the solids AR, LS, &c. have equal bases and altitudes they are equal to one another (6. 12); therefore the solid AC : EG : : no. of parts in AC : no. of parts in EG : : no. of parts in AB : no. of parts in EF : : base AB : base EF. Therefore, rectangular &c. Q. E. D.

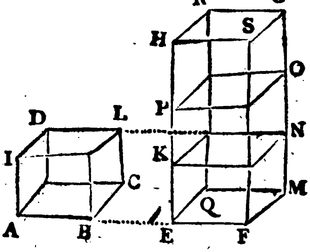
Cor. prisms and cylinders are to one another as their bases. For all prisms, cylinders, and rectangular parallelopipeds of equal bases and altitudes are equal to one another (6. 12 and Cor.).

PROPOSITION VIII. THEOREM.

Rectangular parallelopipeds of equal bases are to one another as their altitudes.

Let AL, EG be two rectangular parallelopipeds on equal bases AC, EM; AL : EG : : alt. AI : EH or MG.

In EH, the greater altitude, take EK = AI, and let a plane pass through KN, making MN = EK. Because the base AC = base EM (by hyp.), and the alt. AI = alt. EK, the solid AL = solid EN (6. 12). Because a parallelopiped is bounded by parallelograms, of which every opposite two are equal, similar, and parallel (Def. 12), any one of those parallelograms may be considered as its base. Hence the solid EN : solid EG : : base FN : base FG, for the altitudes MQ and GR are equal (7. 12). But the parallelogram FN : FG : : base MN : MG, because the altitudes MF, GS are equal (1. 6). Consequently the solid EN : solid EG : : alt. MN : alt. MG (Propor. 34). But the solid AL = EN, and the alt. MN or EK = alt. AI; therefore the solid AL : solid EG : : alt. AI : alt. MG or EH. Therefore, rectangular &c. Q. E. D.



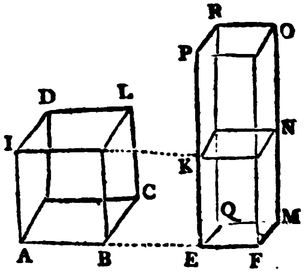
Cor. Prisms and also cylinders on equal bases are to one another as their altitudes. For a rectangular parallelopiped is equal to a prism or cylinder of equal base and altitude (6. 12).

PROPOSITION IX. THEOREM.

The bases and altitudes of equal rectangular parallelopipeds are reciprocally proportional; and if the bases and altitudes of rectangular parallelopipeds be reciprocally proportional, the parallelopipeds are equal to one another.

1. Let AL and EO be two equal rect. parallelopipeds; the base AC : EM : : altitude EP : AI.

Let EN be a rect. parallelopiped on the base EM, and let its altitude EK be equal to the altitude AI of the rect. parallelopiped AL.



Because the altitudes AI and EK are equal, the solid AL : EN : : base AC : EM (7. 12); and because the solid AL = EO, the solid EO : EN : : base AC : EM. But EO : EN : : alt. EP : EK or AI (8. 12); therefore the base AC : EM : : alt. EP : AI (Propor. 34).

2. Let the base AC be to EM as the alt. EP is to AI; the solid AL = EO.

Because the base AC : EM : : alt. EP : AI or EK, and because the base AC : EM : : solid AL : EN (7. 12); the alt. EP : EK or AI : : solid AL : EN (Propor. 34). But the alt. EP : EK : : solid EO : EN (8. 12); therefore AL : EN : : EO : EN, therefore AL = EO (Propor. 32). Therefore, the bases &c. Q. E. D.

Cor. 1. The bases and altitudes of equal prisms, or of equal cylinders, are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the prisms or cylinders are equal.

Cor. 2. The solidity or volume of a rectangular parallelopiped, or of a prism, or a cylinder, is equal to the product of the area of the base multiplied by the altitude.

For it has been proved that the base AC : EM : : alt. EP : AI, therefore AC . AI = EM . EP (Propor. 26); and the solid AL = EO.

PROPOSITION X. THEOREM.

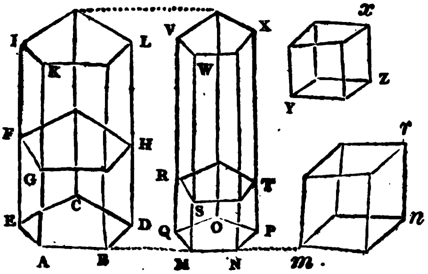
Similar prisms are to one another as the cubes of their like sides.

Let AL and MX be similar prisms, of which the bases are ABDCE and MNPOQ, and the like sides AB, MN: then AL : MX : : cube of AB : cube of MN.

Make AG = AB, and MS = MN; and let FGH, RST be sections parallel to the bases ABDCE, MNPOQ. Let *mr* and *yx* be the cubes of the lines AB and MN; then the bases *mn* and *yz* are the squares of AB and MN.

The prism AL : AH : : alt. AK : AG or AB (Cor. 8. 12), and the prism MX : MT : : alt. MW : MS or MN. The planes KAB, WMN are similar (2 Def. 12), therefore

AK : AB or AG : : MW : MN or MS (1 Def. 6). Consequenty the prism AL : AH : : prism MX : MT (Propor. 34), therefore AL : MX : : AH : MT (Propor. 36).



The base *mn* or AB^2 : ABDCE : : prism or cube *mr* : AH (7. 12 and Cor.), and the base *yz* or MN^2 : MNPOQ : : prism or cube *yx* : MT. But AB^2 : ABDCE : : MN^2 : MNPOQ (1 Cor. 20. 6). Consequently the prism *mr* : AH : : prism *yx* : MT (Propor. 34), therefore *mr* : *yx* : : AH : MT (Propor. 36). Now it has been proved that the prism AL : MX : : prism AH : MT; therefore the cube *mr* : *yx* : : prism AL : MX (Propor. 34). But the cube $mr = AB^3$, and the cube $yx = MN^3$ (by hyp.). Hence the prism AL : MX : : AB^3 : MN^3. Therefore, similar prisms &c. Q. E. D.

Cor. 1. Similar rectangular parallelopipeds are to one another as the cubes of their like sides; and similar cylinders are to one another as the cubes of their diameters.

Cor. 2. Similar prisms, parallelopipeds, and cylinders are to one another as the cubes of their altitudes respectively.

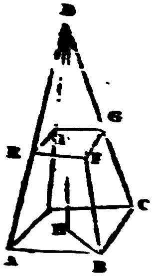
PROPOSITION XI. THEOREM.

If a pyramid be cut by a plane parallel to its base, the section will be similar to the base; and the section and the base will be to each other as the squares of their distances from the vertex of the pyramid.

1. Let ABCD be a pyramid, and EFG a section parallel to the base ABC. From the vertex D draw DI perp. to the section EFG (10. 11), and produce DI to meet ABC in H. The section EFG will be similar to the base ABC; and ABC : EFG : : HD^2 : ID^2.

Join IF and HB. Because the parallel planes ABC, EFG are

cut by the plane DAB, the sections AB and EF are parallel (11. 11): and because the same planes are cut by the plane DBC, the sections BC and FG are parallel. Because the two lines AB, BC, meeting in B, are parallel to the two lines EF, FG, meeting in F, the angle ABC is equal to EFG (9. 11). In the same manner it may be shown that each of the angles in the section EFG is equal to each of the corresponding angles in the base ABC.



Again, since EF and AB are parallel, and in the same plane DAB, the triangles ABD, EFD are similar, therefore DF : DB :: EF : AB (4. 6). For the same reason DF : DB :: FG : BC. Hence EF : AB :: FG : BC (Propor. 34).

It has been proved that the section EFG and the base ABC are equiangular, and have their sides about the equal angles proportional; therefore EFG and ABC are similar (1 Def. 6).

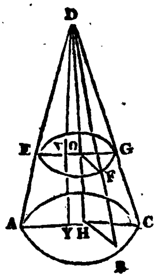
2. Because the lines DH and DB are cut by the parallel planes EFG, ABC, DB : DF : : DH : DI (12. 11). But it has been proved that DB : DF : : AB : EF. Consequently DH : DI : : AB : EF (Propor. 34); therefore $DH^2 : DI^2 :: AB^2 : EF^2$ (Propor. 44). But the base ABC and the section EFG are similar, therefore $AB^2 : EF^2 ::$ ABC : EFG (1 Cor. 20. 6); therefore ABC : EFG :: $DH^2 : DI^2$ (Propor. 34). Now DI is perp. to the plane EFG (by const.), therefore DI is the distance between the vertex D and EFG (Cor. C. 1). But the base ABC is parallel to EFG; therefore it is evident that any two corresponding lines IF, HB in those planes are parallel. Hence DHB is a right angle (Cor. 29. 1), therefore the perp. DH is the distance between D and the base ABC. Therefore, if a pyramid &c. Q. E. D.

PROPOSITION XII. THEOREM.

If a cone be cut by a plane parallel to its base, the section will be a circle; and the section and the base will be to each other as the squares of their distances from the vertex of the cone.

Let ABCD be a cone, and EFG a section parallel to the base ABC; from the vertex D draw DV perp. to EFG (10. 11), and produce it to meet the plane ABC in Y. The section EFG will be a circle, and ABC : EFG :: $DY^2 : DV^2$.

1. Let the planes DHB, DHC pass through DH, the axis of the cone, and meet EFG in the points O, F, G. Because the planes DHB, DHC cut the parallel planes EFG, ABC, the section OF is parallel to HB, and the section OG to HC (11. 11). Since OF is parallel to HB, the angle DOF is equal to DHB (29. 1), and the angle D is common to the triangles DOF, DHB, therefore the triangles are similar. In the same manner it may be proved that the triangles DOG, DHC are similar. Consequently DH : DO : : HB : OF (4. 6), and DH : DO : : HC : OG; therefore HB : OF : : HC : OG (Propor. 34), therefore HB : HC : : OF : OG (Propor. 36). But the radius HB = HC, therefore OF = OG (Propor. 31). In the same manner it may be shown that all lines drawn from O to the perimeter of the section EFG are equal, therefore EFG is a circle.



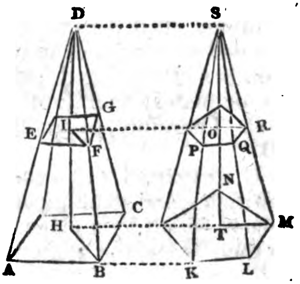
2. Because the lines DY and DH are cut by the parallel planes EFG, ABC, DV : DY : : DO : DH (12. 11); and it has been proved that DO : DH : : OF : HB; therefore DV : DY : : OF : HB (Propor. 34), therefore $DV^2 : DY^2 : : OF^2 : HB^2$ (Propor. 44). But $OF^2 : HB^2$: : circle EFG : circle ABC (R. 6). Consequently the base ABC : section EFG : : $DY^2 : DV^2$, and DY, DV are the distances of the vertex from the planes ABC, EFG, as was shown in the last prop. Therefore, if a cone &c. Q. E. D.

PROPOSITION XIII. THEOREM.

Pyramids of equal bases and altitudes are equal to one another.

Let the pyramids ABCD, KLMS, standing on the same plane, have equal bases ABC, KLMN, and equal altitudes DH, ST; the pyramid ABCD is equal to KLMS.

Let the sections EFG, PQR be parallel to the bases ABC, KLMN respectively; and let the distances DI, SO of the vertexes D, S from the sections be equal. Then ABC : EFG : : DH^2 : DI^2 (13. 12), and KLMN : PQR : : ST^2 : SO^2. But DH = ST, and DI = SO; therefore $DH^2 = ST^2$, and $DI^2 = SO^2$ (Propor. 44); also the base ABC = KLMN, therefore the section EFG = PQR. In the same manner it may be proved that any other two sections at equal distances from the vertexes are equal to each other. But the pyramids are supposed to consist of an infinite number of equal and parallel sections or plates; therefore they are equal. Therefore, pyramids &c. Q. E. D.



COR. If pyramids of equal altitudes, and standing on the same plane, be cut by a plane parallel to their bases, the sections and bases will be proportional.

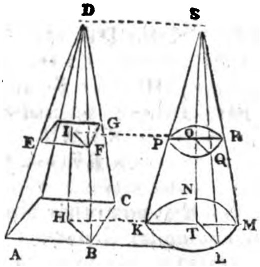
PROPOSITION XIV. THEOREM.

Pyramids and cones of equal bases and altitudes are equal to one another.

Let the pyramid ABCD and the cone KLMS stand on equal bases ABC, KLMN, and have equal altitudes DH, ST; the pyramid is equal to the cone.

Let the sections EFG, PQR, be parallel to the bases ABC, KLMN, respectively; and let DI, SO, the distances of the vertexes from the sections EFG, PQR, be equal. Then ABC : EFG : : DH^2 : DI^2 (11. 12), and KLMN : PQR : : ST^2 : SO^2 (12. 12). But DH = ST, and DI = SO; therefore $DH^2 = ST^2$, and $DI^2 = SO^2$ (Propor. 44). Hence ABC : EFG : : KLMN

: PQR. But ABC = KLMN, therefore EFG = PQR (Propor. 32). In the same manner it may be proved that any two sections at equal distances from the vertexes of the pyramid and cone are equal to each other. But the two solids are supposed to consist of an indefinite number of equal and parallel sections or plates; therefore they are equal. Therefore, pyramids &c Q. E. D.



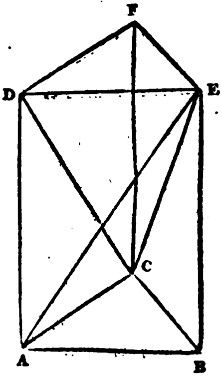
Cor. If a pyramid and a cone of equal altitudes, and standing on the same plane, be cut by a plane parallel to their bases, the sections and the bases will be proportional.

PROPOSITION XV. THEOREM.

Every prism having a triangular base may be divided into three pyramids which have triangular bases, and are equal to one another.*

Let ABC — DEF be a triangular prism; it may be divided into three equal pyramids having triangular bases.

Join AE, EC, CD. Because ABED is a parallelogram, of which AE is the diagonal, the triangle ADE is equal to ABE (34. 1); therefore the pyramid C — ADE is equal to the pyramid C — ABE (13. 12). But the pyramid C — ABE is equal to the pyramid C — DEF, for they have equal bases, ABC, DFE, and the altitude of the prism ABCDEF. Therefore the three pyramids ADEC, ABEC, DFEC are equal to one another. But these pyramids compose the whole prism; therefore the prism ABCDEF is divided into three equal pyramids. Wherefore, every prism &c. Q. E. D.



Cor. 1. From this it is manifest that every pyramid is the third part of a prism of the same base and altitude; for if the base of the prism be any other figure but a triangle, it may be divided into prisms having triangular bases.

* Learners will scarcely understand the demon. of this prop. without models of the three pyramids.

Cor. 2. Pyramids of equal altitudes are to one another as their bases; because the prisms on the same bases as the pyramids, and of the same altitude, are to one another as their bases (Cor. 7. 12).

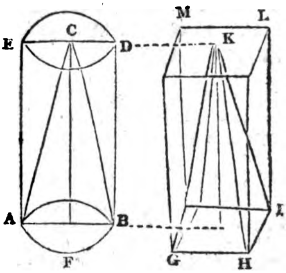
Cor. 3. Similar pyramids are to one another as the cubes of their homologous sides (10. 10).

PROPOSITION XVI. THEOREM.

Every cone is a third part of a cylinder of the same base and altitude.

Let AFBC be a cone, and AFBDE a cylinder of the same base and altitude; the cone is a third part of the cylinder.

Let GHIK be a pyramid, and GHILM a prism on the same base GHI and of the same altitude, and let the base and altitude of the prism be equal to those of the cylinder. Then the cylinder is equal to the prism (6. 12), and the cone is equal to the pyramid (14. 12). But the pyramid GHIK is a third part of the prism GHILM (Cor. 15. 12), therefore the cone AFBC is a third part of the cylinder AFBDE. Therefore, every cone &c. Q. E. D.

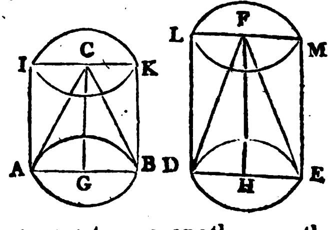


PROPOSITION XVII. THEOREM.

Cones of equal bases are to one another as their altitudes.

Let ABC, DEF be cones of equal bases, and of unequal altitudes CG, FH; the cone ABC : DEF : : alt. CG : FH.

Let ABKI, DEML be cylinders of equal bases, then ABKI : DEML : : alt. CG : FH (Cor.8.12). But the cone ABC is a third of ABKI (16. 12), and the cone DEF is a third of DEML; therefore ABC:DEF : : alt. CG : FH (Propor. 30). Therefore, cones &c. Q. E. D.

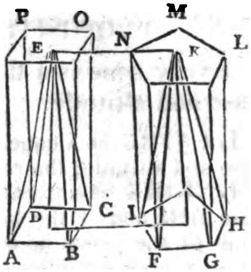


Cor. Pyramids of equal bases are to one another as their altitudes. For cones and pyramids of equal bases and altitudes are equal to one another (14. 12).

PROPOSITION XVIII. THEOREM.

Pyramids of equal altitudes are to one another as their bases.

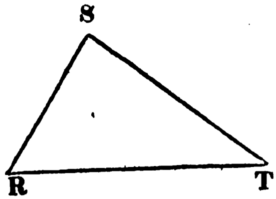
Let ABCE, FGHK be two pyramids having equal altitudes, but unequal bases ABCD, FGHI; the pyramid ABCE : FGHK : : base ABCD : FGHI.



For prisms of equal altitudes are to one another as their bases (Cor. 7. 12), therefore the prism ABCOP : FGHLMN : : base ABCD : FGHI. But a pyramid is one third of a prism of the same base and altitude (Cor. 15. 12); therefore the pyramid ABCE : FGHK : : base ABCD : FGHI. Wherefore, pyramids &c. Q. E. D.

Cor. Cones of equal altitudes are to one another as their bases. For a cone is equal to a pyramid of the same base and altitude (16. 12).

LEMMA. In a right angled triangle if circles, or similar segments of circles, be described on the three sides, the figure described on the hypothenuse is equal to both the figures described on the other two sides.



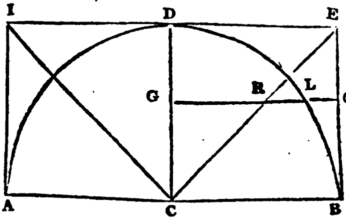
For the fig. on RS : fig. on RT : : $RS^2 : RT^2$ (R. 6),
and the fig on ST : fig. on RT : : $ST^2 : RT^2$;
therefore the fig. on RS : RS^2 : : fig. on RT : RT^2 (Propor. 36),
and the fig. on ST : ST^2 : : fig. on RT : RT^2;
therefore the fig. on RS : RS^2 : : fig. on ST : ST^2 (Propor. 34), therefore the fig on RS + fig. on ST : : $RS^2 + ST^2$: : fig. on RT : RT^2 (Propor. 41). But $RS^2 + ST^2 = RT^2$ (B. 2), therefore the fig. on RS + fig. on ST = fig. on RT (Propor. 32). Therefore, in a right &c. Q. E. D.

PROPOSITION XIX. THEOREM.

Every sphere is two thirds of the circumscribing cylinder; that is, of a cylinder having the same altitude and diameter as the sphere.

Let ADBCA be a hemisphere, and AIDEBCA a cylinder described about it; the hemisphere is equal to two thirds of the circumscribing cylinder.

Let ADB be a semicircle, of which the centre is C; let CD be perp. to AB; let DEBC and DIAC be squares described on DC; draw the diagonal CE, and through any point G in DC draw GO parallel to DE. Let the figure thus constructed revolve about DC; then the square DEBC will describe a cylinder; the sector BCD, which is a quadrant, will describe a hemisphere, of which C is the centre (7 Def.); and the triangle CDE will describe a cone having its vertex at C (11 Def.), and



having for its base the circle described by DE, equal to the circle described by CB, which circle is the base of the hemisphere.

Let L be the point in which GO meets the semicircle ADB, and suppose CL to be joined; then, in the rotation of the plane DEBC on DC the three lines GO, GL, GR, will describe circular sections of the cylinder, hemisphere, and cone respectively. Since CGL is a right angle, the two circles described with the distances CG, GL, are together equal to the circle described with the distance CL or GO (Lemma.).

Now the triangles CDE and CGR are equiangular, therefore CD : CG : : DE : GR. But CD = DE, therefore CG = GR (Propor. 32). Therefore the circles described with the distances GR and GL are together equal to the circle described with the distance GO; that is, the circles described by the revolution of GR and GL about the point G are together equal to the circle described by the revolution of GO about G; or the circular sections of the cone and sphere are together equal to the corresponding section of the cylinder. And because this is the case in every parallel position of GO, it follows that the cylinder AE is equal to the cone CEI and the hemisphere ADBA. But the cone CEI is a third part of the cylinder AE having the same base and altitude (17. 12); consequently the hemisphere ADBA is equal to the remaining two thirds of the cylinder AE. Therefore the whole sphere is equal to two thirds

of twice the cylinder AE, that is, to two thirds of the circumscribing cylinder. Therefore, every sphere &c. Q. E. D.* Ed.

Cor. 1. A cone, a hemisphere, and a cylinder, of the same base and altitude, are to one another as the numbers 1, 2, 3. For it has been proved that a cone is one third (6. 12), and a hemisphere two thirds of a cylinder of the same base and altitude.

Cor. 2. Spheres are to one another as the cubes of their diameters.

For cylinders of the same altitude are to one another as the cubes of their diameters (1 Cor. 10. 12); and a sphere is two thirds of a cylinder of which the diameter and altitude are both equal to the diameter of the sphere. Therefore spheres &c.

* Similar demonstrations of this proposition may be seen in West's Elements of Mathematics, Saunderson's Algebra, and Keith's Euclid.

THE PRINCIPAL THEOREMS IN BOOK XII.

Prisms and cylinders of equal bases and altitudes are all equal to one another.

Prisms and cylinders of equal altitudes are to one another as their bases; and prisms and cylinders of equal bases are to one another as their altitudes.

The bases and altitudes of equal prisms, or of equal cylinders, are reciprocally proportional.

Similar prisms are to one another as the cubes of their like sides; and similar cylinders are to one another as the cubes of their diameters.

Similar prisms, and similar cylinders, are to one another as the cubes of their altitudes respectively.

If a cone, or a pyramid, be cut by a plane parallel to its base, the section will be similar to the base; and the section and the base will be to each other as the squares of their distances from the vertex.

Pyramids and cones of equal bases and altitudes are all equal to one another.

A pyramid is the third part of a prism, and a cone is the third part of a cylinder, of the same base and altitude.

Cones, and also pyramids, of equal altitudes are to one another as their bases.

Cones, and also pyramids, of equal bases are to one another as their altitudes.

Every sphere is two thirds of its circumscribing cylinder.

Spheres are to one another as the cubes of their diameters.

NOTES ON BOOK I.

The first Book of Euclid's Elements contains the principles of all the following Books: it demonstrates some of the most general properties of straight lines, angles, triangles, parallel lines, parallelograms, and other rectilineal figures: it shows the method of constructing certain figures, and of performing different operations.

Some propositions are of little or no use, and may be omitted. Some of these are auxiliary propositions, and were introduced into the Elements merely for the purpose of facilitating the demonstrations of others. Thus, the 16th proposition is implied in the 32nd, and is useless after the 32nd proposition is demonstrated.

The propositions in the first book are not arranged according to the nature of the subjects, but in such order as is adapted to facilitate their demonstrations. By a different arrangement some useless propositions might have been excluded, and perhaps better demonstrations of others might have been given.

DEFINITIONS.

Def. 3. *Line* signifies a stroke, and, in reference to the operation of writing, expresses the boundary or contour of a figure. A straight line has two radical properties, which are distinctly marked in different languages. Firstly, it holds the same undeviating course, and secondly, it traces the shortest distance between its extreme points. The first property is expressed by the word *recta* in Latin, and *droite* in French; and the second property seems to be intimated by the English term *straight,* which is evidently derived from the verb to *stretch.* Accordingly Proclus defines a straight line as stretched between its extremities, and consequently must be the shortest distance between them. LESLIE.

Cor. Def. 7. The equality of all right angles is an obvious inference from the formation and definition of a right angle. M. Legendre makes this corollary a proposition, and demonstrates it by a difficult process, which is scarcely intelligible to learners.

AXIOMS.

"On the principle of congruency Euclid lays down a few simple truths, from which he demonstrates the more complex truths which depend on this principle. Those obvious truths are as follows.

"1. All points coincide.

"2. Straight lines which are equal to one another coincide; and, conversely, straight lines whose extremities coincide are equal.

"3. In two equal angles if the vertexes coincide, and one side of one angle coincide with one side of the other, then the remaining side of the first angle will coincide with the remaining side of the second. Likewise all angles whose sides coincide are equal.

"Though Euclid has not separately enounced those particular axioms subordinate to the general axiom, yet he applies them, as we shall find by analyzing several of his demonstrations."

FENN'S EUCLID.

"That which is here numbered the eighth axiom is not properly an axiom, but a definition. It is the definition of geometrical equality; the fundamental principle on which the comparison of all geometrical magnitudes will be found ultimately to depend.

"The geometrical notions of *equality* and *coincidence* are the same; and even in comparing together spaces of different figures all our conclusions ultimately rest on the imaginary application of one triangle to another; the object of which imaginary application is merely to *identify* the two triangles together in every circumstance connected both with magnitude and figure." STEWART'S PHILOSOPHY.

PROPOSITIONS.

Of two propositions one is contrary, or contradictory to the other, when we deny in one what is affirmed in the other. Thus, two lines cannot be equal and unequal at the same time.

Of two propositions one is the *converse* of the other, when the order of either of them is inverted. Thus, if two sides of a plane triangle be equal, it can be proved that the angles opposite to those sides are equal. Now if we invert the order of this proposition, we shall obtain another proposition which is the converse of it. Thus, if two angles of a plane triangle be equal, it can be proved that the sides opposite to those angles are equal. This proposition is said to be the converse of the former. In the first poposition two sides of a triangle are supposed to be equal, and the equality of the two opposite angles is thence inferred; in the second proposition two angles of

a triangle are supposed to be equal, and the equality of the opposite sides is thence inferred. The two propositions placed in succession may be expressed thus. If two sides of a plane triangle be equal, the angles opposite to those sides are equal; and, conversely, if two angles of a plane triangle be equal, the sides opposite to those angles are equal. Converse propositions are generally true, but not always so. When they are obviously and necessarily true they require no demonstration. Thus, the converse proposition above is sufficiently clear, and needs no proof. The demonstrations are generally indirect, artificial, and scarcely intelligible to students. In some cases it is necessary to apply indirect demonstrations. Thus, it sometimes happens that two magnitudes may be proved to be equal to each other, by showing that if we suppose them unequal, this supposition would involve an absurdity.

PROPOSITION I, II, III.

By a *given finite line* Euclid means a line given both in position and in magnitude. A line may be given in position, but not in magnitude; or it may be given in magnitude (and therefore is finite), but not in position.

In all the propositions in plane geometry Euclid supposes all lines and all parts of figures to be in the same plane.

Euclid's construction and demonstration of the first proposition are tediously minute, and may be expressed as clearly and more concisely as follows. From the centres A and B, with the radius AB, describe two circles cutting each other in some point C. From C draw CA, CB; then ABC is the triangle required. For AC is equal to AB, because they are radii of the same circle (20 Def), and BC is equal to AB. Therefore AC is equal to BC (1 Ax.). Therefore the three sides AB, AC, BC are equal to one another; therefore ABC is an equilateral triangle.

Modern geometricians generally assume the second and third propositions as postulates, or evident principles, which require no proof. Euclid's demonstration of the second proposition is artificial, and somewhat obscure to students. The demonstration of the third prop. depends on the second, and therefore is equally exceptionable. Euclid's solutions of those problems are so like pedantic trifling that I have retained them with reluctance, and have given others in the Notes, which are practical and modern.

The following demonstrations involve the idea of motion, or the translation of a magnitude from one place to another. Euclid employs motion in some of his demonstrations; and modern geometricians admit it in all parts of mathematics.

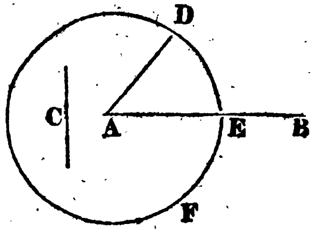
Any assumptions or constructions may be employed in demonstrations which are *manifestly* possible, or have been already shown to be possible.

The following observations are taken from Cresswell's Geometry, Preface, page 19.

"We readily perceive when we consider the nature of continued quantity that in any given finite line there is a point which divides it into two equal parts; and it follows from the definition of a circle that every given circle has a centre: so that in reasoning about lines and circles we always assume those points whithout any impropriety. In developing the theory of geometry we may also show the possible existence of a point, a line, or a surface, which shall have some particular position; or, in the case of a line, or a surface, some assigned magnitude. This, indeed, is the most proper mode of proceeding while we are pursuing the course of a science of pure reasoning."

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.



Let A be the given point, and C the given straight line, it is required to draw from A a straight line equal to C.

From the centre A, with the radius C, describe a circle, and from A to any point D in the circumference draw a line AD; then AD is equal to C (20 Def.).

PROPOSITION III. PROBLEM

From the greater of two given straight lines to cut off a part equal to the less.—*(See last figure.)*

Let AB and C be the given straight lines, of which AB is the greater. It is required to cut off from AB a part equal to C.

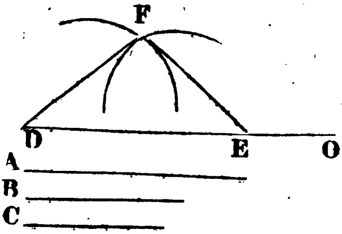
From the centre A, with the radius C, describe a circle cutting AB in E (3. post.), then AE is a part cut off equal to C (20 Def.).

PROPOSITION XXII. PROBLEM.

To construct a plane triangle of which the three sides are given.

Let the stright lines A, B, C represent the three sides of the triangle. It is required to construct a triangle of which the three sides shall be equal to A, B, C, each to each.

From any point D draw an indefinite straight line DO, and to it apply the longest side A, extending from D to E. From the centre D, with a radius equal to B, describe a circle, and from the centre E, with a radius equal to C, describe another circle. Then, because the sides B and C together are greater than A (20. 1), the circles will cut each other in some point F. Draw the straight lines DF and EF. Then DEF is the triangle required.



For DE is equal to A by construction, and because all the radii of the same circle are equal (20 Def.) DF is equal to B, and EF is equal to C. Therefore the three sides of the triangle DEF are equal to the three given straight lines A, B, C, each to each.

Cor. If the sides B and C be equal the triangle will be isosceles; and if all the sides be equal to one another the triangle will be equilateral, and the problem becomes the same as Euclid's first proposition.

NOTES ON BOOK II.

This book treats of the segments of lines and of the rectangles and squares contained by the parts of a straight line divided in a certain manner, and proves that they are equal to other rectangles or squares contained by the parts of the same line divided in a different manner; it also treats of the equality of rectangles and squares contained by the sides, and parts of the sides, of triangles.

If lines and numbers be divided into parts in the same manner, the properties of the parts of both are the same, and may be easily and concisely demonstrated by the first principles of algebra.

Some of the propositions in this book are of no practicl use, and are not referred to in the demonstrations of subsequent pro-

positions. The geometrical demonstrations of them are tedious or difficult. Hence, according to the example of recent writers, it was deemed expedient to omit them in this work.

In the second book some common algebraical signs have been introduced for the sake of representing more concisely and clearly the addition and subtraction of the rectangles on which the demonstrations depend. The concise language of algebra brings the steps of the reasoning nearer to one another, and the force of the whole is more clearly and directly perceived by the reader. Some of Euclid's demonstrations have been changed for others which are shorter and plainer.

As some writers and teachers, particularly of the old school of mathematics, object to the use of symbols in geometry, and still adhere to the verbose and tedious style of Dr. Simson's translation of Euclid's Elements, I annex a short extract from the preface to Elements of Geometry by D. Cresswell, M. A. Fellow of Trinity College, Cambridge. Mr. Cresswell recommends the use of symbols in geometry from his own experience.

"Throughout the following work certain symbols are used for the sake of conciseness, many of which are borrowed from the language of universal arithmetic. They are merely abbreviated representations of the words, and the phrases, for which they are severally put. Whithout affecting the nature of the demonstrations they render the force of them more easily and more quickly perceived. This plan is followed by most of the foreign mathematicians who have written on the subject of geometry."

Playfair expresses the same sentiment.

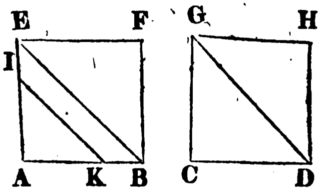
In the demonstrations of certain propositions Euclid assumes two principles which appear to be sufficiently evident.

1. Squares described on equal straight lines are equal to one another. *See Cor.* 2. *F.* 1.

2. The sides of equal squares are equal to one another.

These two propositions may be easily demonstrated as follows.

1. Let AF, CH be two squares described on equal straight lines AB, CD. The squares AF, CH are equal.



Draw the diagonals BE, DG. Because the two sides AE, AB are equal to the two sides CG, CD, each to each (by hyp.), and the angle BAE is equal to DCG, the triangles BAE, DCG are equal. But the triangle BAE is half of the paral. AF (34. 1), and the triangle DCG is half of the paral. CH. Consequently the parals. AF, CH are equal (6 Ax.) Therefore, squares &c. Q. E. D.

Cor. Parallelograms which have two sides of one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, are equal in all respects.

2. Let the squares AF, CH be equal to each other; the sides AB, CD are equal.

If AB, CD be not equal, one of them is greater than the other. Let AB be the greater, and make AK, AI each equal to CD. Because AK, AI are each equal to CD, and CD is equal to CG, AI is equal to CG; therefore the two sides AK, AI are equal to the two sides CD, CG each to each; and the angles A and C are equal; therefore the triangles IAK, GCD are equal. The triangles GCD, EAB are the halves of the equal squares CH, AF, therefore they are equal. Hence the triangle IAK is equal to EAB, the less to the greater, which is impossible. Therefore the side AB is not greater than CD. In the same manner it may be proved that AB is not less than CD. Consequently AB is equal to CD. Therefore, the sides &c. Q. E. D.

Cor. The square of a greater line is greater than the square of a less line.

The following curious proposition may be easily demonstrated by means of the figure of Prop. B. 2.

In a right angled triangle ABC if the sides DA, EB, of the square AE described on the hypothenuse AB, be produced to meet the sides (produced if necessary) of the squares described on the two sides AC, BC, in K and N, the triangles AFK, BIN, cut off by them will be equal and equiangular to the given triangle.

From the right angles BAK, CAF take the common angle CAK, then the angle BAC = FAK. The right angle AFK = ACB, and the side AF = AC. Consequently the triangles AFK, ABC are equal in all respects (A. 1).

Again, from the right angles CBI, ABN take the common angle CBN, then the angle IBN = ABC. The right angle I = ACB, and the side BI = BC. Consequently the triangles IBN ABC are equal in all respects. Therefore, in a right &c. Q. E.D.

NOTES ON BOOK. III

This book contains the fundamental properties of circles, respecting lines, angles, and figures inscribed in them, and lines cutting or touching them, &c.

Many propositions in this book have been omitted, because they are seldom or never read in places of education. Some useful elementary propositions, found in modern books of geometry, have been added; and some demonstrations of certain propositions, which are shorter or plainer than Euclid's, have been substituted for them.

PROPOSITION I.

There is a defect in Euclid's demonstration of this proposition which has escaped the notice of Simson and Playfair. For if any point H be taken in CD, and Euclid's demonstration be applied, word for word, the angles FDB, HDB will be equal, and therefore no absurdity will follow. the demonstration has been corrected according to Fenn's Euclid.

NOTES ON BOOK IV.

This book treats solely of the description of regular polygons in and about a circle, and the description of a circle in and about regular polygons. The subjects of it make a part of practical geometry, to which it properly belongs. It is not read in the English universities, nor, I believe, in any of ours.

NOTES ON BOOK VI.

The principal object of Book VI is to apply the doctrine of ratio and proportion to lines, angles, rectilineal figures, and circles. This book treats of the ratios of similar rectilineal figures, of angles and their opposite arches, of the diameters and circumferences of circles, and their diameters and surfaces. The propositions in this book constitute the most important part of plane geometry, for they are applicable to all the various departments of mathematics in which geometry is concerned.

PROPOSITION I.

If there be any objection to the latter part of the demonstration of this proposition, the following alteration may be substituted in line 5, bottom of page 110.

The triangle ACH : triangle ADL : : no. of equal parts in ACH : no. of equal parts in ADL. But the no. of parts in ACH : no. of parts in ADL : : no. of parts in the base CH : no. of parts in DL : : base CH : DL. Consequently the triangle ACH : ADL : : base CH : DL (Propor. 34). Therefore, triangles &c. Q. E. D.

The two demonstrations of this proposition do not include the case of incommensurable quantities, which kind of quantities may, without impropriety, be called incomprehensible quantities. The reason of this defect, if it be deemed such, is, that I know no demonstration of that case which is not incorrect, or unintelligible, or unsatisfactory.

PROPOSITIONS XI, XII, XIII.

These three problems may be resolved algebraically as follows.

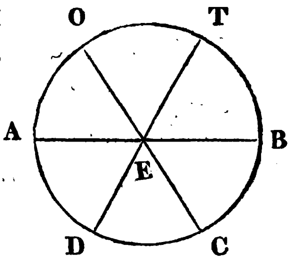
Prop. XI. Let AB $= a$, BD $= b$, BC $= x$; then $a : b :: b : x$, $\therefore x = \frac{bb}{a}$.

Prop. XII. Let a, b, c, denote the three given lines, and let $x =$ HF; then $a : b :: c : x$, $\therefore x = \frac{bc}{a}$.

Prop. XIII. Let AB $= a$, BC $= b$, and $x =$ mean required; then $a : x :: x : b$, $\therefore xx = ab$, $\therefore x = \sqrt{ab} =$ DB.

PROPOSITION XXXIII.

Several demonstrations of this proposition, including both commensurable and incommensurable quantities, have been attempted by different authors; but they are all either unsatisfactory, or unintelligible to learners. The proposition may be proved in a plain and satisfactory manner as follows.



Suppose two lines to be placed on each other, so as to make only one line AB. Let one line be fixed, and the other moveable round the centre E of a circle. As soon as AE begins to move round the centre E, it will describe arches AO, OT, &c. and will make angles with the fixed line.

The successive angles AEO, OET, &c. will be greater or less according as the opposite arches are greater or less. When the point A arrives at T the angle AET will be described, and when it arrives at B the arch ATB will be a semicircle, or 180 degrees; and the sum of the angles, or AEO + OET + TEB, is equal to two right angles (13. 1). Let EA continue to revolve, and the point A will describe the arches BC, CD, DA; and the radius EA will make the angles BEC, CED, DEA, in the same time. Hence it appears that the arches and angles begin and end together. Consequently the successive angles vary as the opposite arches; therefore the arches are the measures of the angles; that is, the angles are to one another as the arches which are opposite to them, or the angle AEO : AET :: arch AO : AT, or the angle AEO : OET :: arch AO : OT.

This is 1 Cor. 33, and is manifestly the same as if the angles AEO, AET, &c. were at the centres of two equal circles.

PROPOSITIONS Q, R, S.

These three propositions are demonstrated by the method of Indivisibles, which is explained in the Note on 6. 12, page 185.

NOTES ON BOOK XI.

This book treats of the intersection of straight lines with planes, and of planes with planes.

Most of the propositions in Playfair's second Supplement are so plain and obvious that they are nearly self-evident, and scarcely require any proof. Indeed they are so simple in their nature that many of the demonstrations are artificial and difficult, and do not render their truth more manifest to the reader. As they do not often occur in mathematics, students might be permitted to omit the demonstrations; and it would perhaps be sufficient to illustrate the propositions by simple experiments made by cutting card paper (or any thick paper), and disposing the parts in such positions as the enunciations indicate.

NOTES ON BOOK XII.

This book treats of the properties and relations of solid bodies.

Most of the propositions are taken from Keith's Euclid, which contains a better selection of theorems than Playfair's Geometry. Besides, the demonstrations are free from the obscurity of Euclid's method of demonstration and uncouth phraseology, which seem to have impeded the study of geometry in Britain and America. The use of the geometry of solids seldom occurs, and therefore this Book need not be read till it is wanted in the theory of Mensuration of solids.

PROPOSITION VI.

This proposition and others are demonstrated by the Method of Indivisibles, which was invented by Cavalleri, an Italian, and published in 1635 This method is not strictly geometrical, but the demonstrations derived from it are plain and satisfactory to students. By it the areas of plane and curve surfaces, and the contents of solids are determined with ease and accuracy.

It is applied by recent writers in mathematical demonstrations without any explanation of its principles, which are plain and simple, and may be explained as follows

1. The limit of any line, when it is diminished, is a line infinitely short, or a point. For by continually taking away parts of the line we can render the remainder shorter than any assignable line. In the same manner the limit of a surface is

a surface infinitely narrow, or a line; and the limit of a solid is a solid infinitely thin, or a surface.

2. It follows that a line is equal to an infinite number of points. For the line being divided into parts, the whole line is equal to the sum of all its parts. But as we diminish the magnitude of each part, or increase the number of parts, the limit of each part is a point. Hence the line will be equal to the sum of those points, which are infinite in number. In the same manner a surface is equal to an infinite number of lines, and a solid to an infinite number of surfaces. Consequently a point is naturally called the *element* of a line, and a line the element of a surface, and a surface the element of a solid. Each of these elements, having at least one of its dimensions infinitely small, and therefore is incapable of diminution, or subdivision, is called an *indivisible.*

3. There are various ways in which this conception may be applied to the same geometrical quantity. Thus, a circle may be conceived to be composed either of parallel chords, or of the circumferences of concentric circles. A solid may be conceived to consist of parallel *laminæ*, or extremely thin plates, resembling the leaves of a book, but the thickness of a leaf is infinitely greater than that of the element of the solid.

4. The difference between a chord and its arch can be made as small as we please in comparison of either of them; so that either of them may be called the limit of the other. In this sense an infinitely small arch is equal to its chord; and any finite arch is equal to an endless number of straight lines. In the same manner any infinitely small portion of a curve surface is equal to the plane which forms its base; and any finite portion of a curve surface is equal to an endless number of planes. Hence a circle will be a regular polygon of an infinite number of sides, and a sphere will be a polyedron of an infinite number of faces.

Without conceiving bodies to be composed of an infinite number of extremely thin elementary plates, parallel to one another, their equality may be proved by help of the following principle, which is so obvious that it may be admitted as an axiom.

Axiom. If any two solids standing on the same base, or on equal bases, and between the same parallels, be cut by numerous planes in directions parallel to their bases, and if all the sections at equal altitudes be equal to one another, the two solids will be equal to each other.

From this axiom we infer that the proposition is true.

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